NEW CONGRUENCES MODULO 3 FOR 11-CORE PARTITION FUNCTION

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Abstract: If t is a positive integer, then a partition of a nonnegative integer n is a t-core if none of the hook numbers of the associated Ferrers-Young diagram is a multiple of t. These partitions play important roles in the study of irreducible representations of the symmetric group S_n. Let $a_t(n)$ denote the number of t-core partitions of n. There are many interesting congruences proved by Granville, Ono, Garvan for the p-core partition functions $a_p(n)$ where p is a prime. In this paper, we derive new congruences modulo 3 for the 11-core partition function $a_{11}(n)$ by using an identity given by Chan and Toh and a p-dissection of Ramanujan's theta function f_1 due to Cui and Gu.

Keywords: Congruences, Dissection, t-core partitions, Theta function.

I. INTRODUCTION AND STATEMENT OF RESULTS

A partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Let $p(n)$ be the number of partitions of n, for example $p(5)=7$, the seven partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

If $\Lambda=(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of n, then the corresponding Ferrers-Young diagram of Λ is the following staircase arrangement of nodes with λ_i nodes in the i-th row:

● ● ·····● ● λ_1 nodes
●● ·····● λ_2 nodes
· ·
· ·
●● ····· ● λ_k nodes

Ferrers-Young diagram of Λ

Label the nodes (i, j), as they are matrix entries. We associate a geometric object called a hook to each node. The (i, j) hook is the set of nodes directly to the right of the specified node, the nodes directly below it, and the node itself. If λ'_j denotes the number of nodes in column j, the hook number $H(i,j)$ of the (i, j) node is defined by $H(i,j) := \lambda_i + \lambda'_j - i - j + 1$.

Definition: A t-core partition of n is a partition of n in which none of the hook numbers is a multiple of t. Let $a_t(n)$ denote the number of t-core partitions of n.

Example: The Ferrers-Young diagram of the partition $\Lambda=(4,3,1)$ of 8 is

1 2 3 4
1● ● ●●
2 ●● ●
3 ●

where, $\lambda_1=4, \lambda_2=3, \lambda_3=1, \lambda'_1=3, \lambda'_2=2, \lambda'_3=2, \lambda'_4=1$ and hence $H(1, 1)=6$, $H(1, 2)=4$, $H(1, 3)=3$, $H(1, 4)=1$, $H(2, 1)=4$, $H(2, 2)=2$, $H(2,3)=1$, $H(3, 1)=1$. Hence the partition $\Lambda=(4,3,1)$ of 8 is t-core if $t=5$ or $t \geq 7$.

The number of irreducible representations of the symmetric group S_n equals $p(n)$, the number of partitions of n. The generating function for $p(n)$ is given by,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1},$$

whereas customary, we define $f_k=(q^k;q^k)_\infty=\prod_{m=1}^{\infty}(1-q^{km})$.

Garvan, Kim and Stanton [7] showed,

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)^t_\infty}{(q;q)_\infty} = \frac{{f_t}^t}{f_1}. \qquad (1.1)$$

Using crank and t-core partitions they also produced more combinatorial explanations for Ramanujan's famous congruences,
$p(5n + 4) \equiv 0 \pmod 5$,
$p(7n + 5) \equiv 0 \pmod 7$,
$p(11n + 6) \equiv 0 \pmod{11}$.

When t is a power of 2, arithmetic properties of $a_t(n)$ have been proved by Boylan and Chen [2,4]. For 4-core, 8-core and 16-core partition functions, Hirschhorn, Kolitsch and Sellers, Ono an Sze proved a family of interesting congruences. [2, 9, 10, 11, 13, 14]. When t is a prime p, then $a_p(n)$ equals the number of irreducible representations of the symmetric group S_n over a field of characteristic p which are projective S_n-modules. For more information (see, [8,12]). There are many interesting congruences for $a_p(n)$ proved by [6]. Recently, Radu and Sellers [15] proved other Garvan-type congruences namely $a_7(14n+7,9,13) \equiv 0 \pmod 8$ and $a_{11}(22n+2,8,12,14,16) \equiv 0 \pmod 2$. The aim of this

paper is to prove new congruences for $a_{11}(n)$. The following is our main theorem.

1.1. Theorem: For all $n\geq 0$,
$a_{11}(33n + 11, 20, 26, 29, 32) \equiv 0 \pmod 3$. (1.2)
In order to prove our main result, we collect a few definitions and Lemmas in section 2. In section 3 we prove our main results.

II. PRELIMINARIESE

In this section we first restate some definitions and lemmas which are helpful in proving our main results.
For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as, see [1].

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}. \quad (2.1)$$

The most important special cases of $f(a, b)$ are

$$\Phi(q):=f(q,q)=1+2\sum_{n=1}^{\infty} q^{n^2}$$
$$=(-q;\, q^2)_\infty^{\,2}(q^2;\, q^2)_\infty=\frac{f_2^5}{f_1^2 f_4^2}, \quad (2.2)$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}$$

$$=\frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty}=\frac{f_2^2}{f_1}, \quad (2.3)$$

and

$$f(-q):=f(-q,-q^2)$$

$$=\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty=f_1. \quad (2.4)$$

From the Binomial Theorem, for any positive integer, k,

$$f_k^3 \equiv f_{3k} \pmod 3. \quad (2.5)$$

2.1 Lemma (Chan and Toh [3, Eqn. (1.19)])

$$\sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n + 2)q^n = 2\frac{f_3 f_{33}}{f_1^2 f_{11}^2} + 3q\frac{f_3^2 f_{33}^2}{f_1^3 f_{11}^3}$$
$$+3q^2\frac{f_3^3 f_{33}^3}{f_1^4 f_{11}^4}, \quad (2.6)$$

where $\sum_{n=0}^{\infty} p_{[1^1 11^1]}(n)q^n = \frac{1}{f_1 f_{11}}$.

2.2 Lemma (Cui and Gu [5, Theorem 2.2])
If $p\geq 5$ is a prime and

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6 \end{cases}$$

$$(q;q)_\infty =$$

$$\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} \left(q^{p^2};q^{p^2}\right)_\infty. \quad (2.7)$$

Furthermore, if $\frac{p-1}{6} \leq k \leq \frac{p-1}{6}$, $k \neq \frac{\pm p-1}{6}$ *then* $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{24}$.

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Taking $t=11$ in (1.1), we have

$$\sum_{n=0}^{\infty} a_{11}(n)q^n = \frac{f_{11}^{11}}{f_1}. \quad (3.1)$$

In view of(2.5),we can rewrite (3.1) as follows

$$\sum_{n=0}^{\infty} a_{11}(n)q^n = \frac{f_{11}^{11}}{f_1}$$
$$=\frac{f_{11}^{12}}{f_1 f_{11}}\equiv\frac{f_{33}^4}{f_1 f_{11}} \pmod 3.$$

The above congruence implies that,

$$\sum_{n=0}^{\infty} a_{11}(n)q^n \equiv f_{33}^4 \sum_{n=0}^{\infty} p_{[1^1 11^1]}(n)q^n,$$

From the above equation extracting the coefficients of the form q^{3n+2}we have,

$$\sum_{n=0}^{\infty} a_{11}(3n + 2)\, q^{3n+2}$$
$$\equiv f_{33}^4 \sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n + 2)q^{3n+2} \pmod 3. \quad (3.2)$$

Dividing both sides of (3.2)by q^2 and then replacing q^3 by $q^{\frac{1}{3}}$, we have

$$\sum_{n=0}^{\infty} a_{11}(3n + 2)\, q^n$$
$$\equiv f_{11}^4 \sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n + 2)q^n \pmod 3. \quad (3.3)$$

Using (2.5) we can rewrite (2.6) as follows,

$$\sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n + 2)q^n \equiv f_1 f_{11} \pmod 3. \quad (3.4)$$

Substituting the identity (3.4) in (3.3), we have

$$\sum_{n=0}^{\infty} a_{11}(3n + 2)\, q^n \equiv f_1 f_{11}^5 \pmod 3. \quad (3.5)$$

Taking $p=11$ in (2.7)and then substituting it into (3.5), we obtain

$$\sum_{n=0}^{\infty} a_{11}(3n + 2)\, q^n \equiv f_{11}^5 \times$$

$\left(\sum_{\substack{k=-5 \\ k\neq -2}}^{5}(-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{11(17+3k)}, -q^{11(16-3k)}\right) + q5q121;q121\infty \pmod 3\right).(3.6)$

Where $-5 \leq k \leq 5$, $k \neq -2$ then $\frac{3k^2+k}{2} \not\equiv 5 \pmod{24}$.

It is easily checked that for any $-5 \leq k \leq 5, k \neq -2$,

$$\frac{3k^2+k}{2} \not\equiv 3,6,8,9,10 \pmod{11}. \quad (3.7)$$

Using (3.7)and comparing the coefficients of the form $q^{11n+j}, j \in \{3,6,8,9,10\}$, from both sides of (3.6),we arrive at (1.2).

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