A GENERALIZATION OF MIDPOINT TYPE INEQUALITY OF M-LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract- In this study, we establish generalized new inequalities for differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality by using m - $\log$ -convex function.The results presented here would provide extensions of those given in earlier works.

Keywords- Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, integral inequalities.

I. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [4]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \quad (1.1)$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. A function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if whenever $x,y\in a,b]$ and $t\in[0,1]$, the following inequality holds

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

Definition 1 The function $f:[a,b]\subset\mathbb{R}\to(0,\infty)$ is said to be a logarithmically convex if the following inequality holds

$$\left|f(xt+(1-t)y)\right|\leq\left[\left|f(x)\right|\right]^t\left[\left|f(y)\right|\right]^{(1-t)} \quad (1.2)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics ([2], [4]). We note that if f and g are convex functions and g is monotonic nondecreasing, then gof is convex. Moreover, since $f=exp(logf)$, it follows that a log-convex function is convex, but the converse is not true (see, [4]). This fact is obvious from (1.2) as by the arithmetic-geometric mean inequality, we have

$$\left[\left|f(x)\right|\right]^t\left[\left|f(y)\right|\right]^{(1-t)}\leq tf(x)+(1-t)f(y).$$

for all $x,y\in I$ and $t\in 0,1]$.

Definition 2 The function $f:[0,b]\to(0,\infty)$ is said to be $m-$logarithmically convex if the inequality

$$\left|f(xt+m(1-t)y)\right|\leq\left[\left|f(x)\right|\right]^t\left[\left|f(y)\right|\right]^{m(1-t)}$$

holds for all $x,y\in[0,b],m\in(0,1]$ and $t\in[0,1]$.

Clearly, if putting $m=1$ in Definition 2, then f is just the ordinary logarithmically convex function on $[0,b]$.

For several recent results concerning Hermite-Hadamard integral inequality, we refer the reader to ([1]-[11]).

In [3], Kiris and Sarikaya gave the following important lemma:

Lemma 1 Let $f:I^o\subset\mathbb{R}\to\mathbb{R}$, be a differentiable mapping on $I^o,a,b\in I^o$ (I^o is the inteior of I) with $a<b$. If $f'\in L([a,b])$, then we have

$$\frac{v}{b-a}\int_a^{\frac{a+b}{2}}f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx-\frac{\mu+v}{2}f\left(\frac{a+b}{2}\right)$$
$$=(b-a)\left[\int_0^{\frac{1}{2}}\mu tf'(at+(1-t)b)dt+\int_{\frac{1}{2}}^1(vt-v)f'(at+(1-t)b)dt\right]$$

where $\mu,v>0$.

The goal of this study is to establish some generalized integral inequalities of Hermite-Hadamard type for $m-$log-convex function.

II. MAIN RESULTS

Now we will give some integral inequalities of Hermite-Hadamard type for $m-$ log-convex functions. We will establish some new results connected with the right-hand side of (1.1) by used the Lemma 1.

Theorem 1 Let $f:I^o\subset\mathbb{R}\to\mathbb{R}$, be a differentiable mapping on $I^o,a,b\in I^o$ (I^o is the inteior of I) with $0\leq a<b$ and $f'\in L([a,b])$. If $\left|f'\right|$ is $m-$ logarithmically convex, then we have

$$\left|\frac{v}{b-a}\int_a^{\frac{a+b}{2}}f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx-\frac{\mu+v}{2}f\left(\frac{a+b}{2}\right)\right|$$

$$\le (b-a)\left\{\left[\frac{\mu\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]}-\frac{\mu\left(\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}-\left|f'\left(\frac{b}{m}\right)\right|^m\right)}{\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]^2}\right]+\left[\frac{v\left(\left|f'(a)\right|-\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}\right)}{\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]^2}-\frac{v\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]}\right]\right\}$$

where $\mu, v > 0$ and $m \in (0,b]$.

Proof. From Lemma 1 and using the fact that $\left|f'\right|$ is m – logarithmically convex on $[a,b]$, we have

$$\left|\frac{v}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx-\frac{\mu+v}{2}f\left(\frac{a+b}{2}\right)\right|$$

$$\le (b-a)\left\{\int_0^{\frac{1}{2}}\mu t\left|f'(at+m(1-t)\frac{b}{m})\right|dt+\int_{\frac{1}{2}}^1(v-vt)\left|f'(at+m(1-t)\frac{b}{m})\right|dt\right\}$$

$$\le (b-a)\left\{\int_0^{\frac{1}{2}}\mu t\left[\left|f'(a)\right|^t\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t)}\right]dt+\int_{\frac{1}{2}}^1(v-vt)\left[\left|f'(a)\right|^t\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t)}\right]dt\right\}$$

$$= (b-a)\left\{\mu\left|f'\left(\frac{b}{m}\right)\right|^m\int_0^{\frac{1}{2}}t\left[\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^t dt+v\left|f'\left(\frac{b}{m}\right)\right|^m\int_{\frac{1}{2}}^1(1-t)\left[\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^t dt\right\}$$

$$= (b-a)\left\{\left[\frac{\mu\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]}-\frac{\mu\left(\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}-\left|f'\left(\frac{b}{m}\right)\right|^m\right)}{\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]^2}\right]+\left[\frac{v\left(\left|f'(a)\right|-\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}\right)}{\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]^2}-\frac{v\sqrt{\left|f'\left(\frac{b}{m}\right)\right|^m\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-m\ln\left(\left|f'\left(\frac{b}{m}\right)\right|\right)\right]}\right]\right\}$$

which completes the proof.

Remark 1 If we choose $m=1$ in Theorem 1, then we have

$$\left|\frac{v}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx-\frac{\mu+v}{2}f\left(\frac{a+b}{2}\right)\right|$$

$$\le (b-a)\left\{\left[\frac{\mu\sqrt{\left|f'(b)\right|\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-\ln\left(\left|f'(b)\right|\right)\right]}-\frac{\mu\left(\sqrt{\left|f'(b)\right|\left|f'(a)\right|}-\left|f'(b)\right|\right)}{\left[\ln\left(\left|f'(a)\right|\right)-\ln\left(\left|f'(b)\right|\right)\right]^2}+\left[\frac{v\left(\left|f'(a)\right|-\sqrt{\left|f'(b)\right|\left|f'(a)\right|}\right)}{\left[\ln\left(\left|f'(a)\right|\right)-\ln\left(\left|f'(b)\right|\right)\right]^2}-\frac{v\sqrt{\left|f'(b)\right|\left|f'(a)\right|}}{2\left[\ln\left(\left|f'(a)\right|\right)-\ln\left(\left|f'(b)\right|\right)\right]}\right]\right\}$$

which is proved by Sarikaya et al. in [12].

Theorem 2 Let $f: I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a,b \in I^o$ with $0 \le a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^q$ is $m-$ logarithmically convex $q>1$, then we have

$$\left|\frac{\nu}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)dx-\frac{\mu+\nu}{2}\,f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)\left|f'\left(\frac{b}{m}\right)\right|^m}{2^{\frac{p+1}{p}}\,q^{\frac{1}{q}}(p+1)^{\frac{1}{p}}\left(\ln\left(\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right)\right)}\times$$

$$\left\{\mu\left(\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|^{\frac{q}{2}}-1\right)^{\frac{1}{q}}+\nu\left(\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|^{q}-\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|^{\frac{q}{2}}\right)^{\frac{1}{q}}\right\}$$

where $\mu,\nu>0$ and $\frac{1}{p}+\frac{1}{q}=1, m\in(0,b]$.

Proof. From Lemma 1 and using Hölder's inequality, we have

$$\left|\frac{\nu}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)dx-\frac{\mu+\nu}{2}\,f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}}(\mu t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^{1}(\nu-\nu t)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is $m-$logarithmically convex, we get

$$\left|\frac{\nu}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)dx-\frac{\mu+\nu}{2}\,f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)$$

$$\left\{\left(\frac{\mu^p}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\left|f'\left(\frac{b}{m}\right)\right|^{qm}\left[\int_0^{\frac{1}{2}}\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|^{tq}dt\right]\right)^{\frac{1}{q}}+\right.$$

$$\left.\left(\frac{\nu^p}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\left|f'\left(\frac{b}{m}\right)\right|^{qm}\left[\int_{\frac{1}{2}}^{1}\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|^{tq}dt\right]\right)^{\frac{1}{q}}\right\}$$

$$=\frac{\left|f'\left(\frac{b}{m}\right)\right|^m(b-a)}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}\left(\ln\left|\frac{f'(a)}{\left(f'\left(\frac{b}{m}\right)\right)^m}\right|\right)}$$

$$\times\left[\nu\left(\frac{|f'(a)|^q}{\left|f'\left(\frac{b}{m}\right)\right|^{mq}}-\frac{|f'(a)|^{\frac{q}{2}}}{\left|f'\left(\frac{b}{m}\right)\right|^{\frac{mq}{2}}}\right)^{\frac{1}{q}}+\mu\left(\frac{|f'(a)|^{\frac{q}{2}}}{\left|f'\left(\frac{b}{m}\right)\right|^{\frac{mq}{2}}}-1\right)^{\frac{1}{q}}\right]$$

which completes the proof.

Remark 2 If we choose $m=1$ in Theorem 2, then we have

$$\left|\frac{\nu}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx+\frac{\mu}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)dx-\frac{\mu+\nu}{2}\,f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{2^{\frac{p+1}{p}}q^{\frac{1}{q}}(p+1)^{\frac{1}{p}}\left(\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{\frac{1}{q}}}\times$$

$$\left[\nu\left(|f'(a)|^q-\sqrt{|f'(a)|^q|f'(b)|^q}\right)^{\frac{1}{q}}+\right.$$

$$\left.\mu\left(\sqrt{|f'(a)|^q|f'(b)|^q}-|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

which is proved by Sarikaya et al. in [12].

Remark 3 Under assumptation of Remark 2 with $\mu=\nu$. Then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}\left(\ln\left|\frac{f'(a)}{f'(b)}\right|\right)^{\frac{1}{q}}}\times$$

$$\left[\left(|f'(a)|^q-\sqrt{|f'(a)|^q|f'(b)|^q}\right)^{\frac{1}{q}}+\right.$$

$$\left.\left(\sqrt{|f'(a)|^q|f'(b)|^q}-|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

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