ON COMPARE SOME INTEGRAL INEQUALITIES ASSOCIATED WITH FRACTIONAL INTEGRALS

[1]MEHMET ZEKI SARIKAYA, [2]FUAT USTA, [3]HÜSEYIN BUDAK

[1,2,3] Department of Mathematics, Faculty of Science and Letter, Düzce University, TURKEY
E-mail: [1]sarikayamz@gmail.com, [2]fuatusta@duzce.edu.tr, [3]hsyn.budak@gmail.com

Abstract- In this study, we firstly give two general integral identities fractional integral operators utilizing a kernel contains two parameters. Secondly, we use this integral identities to derive some new generalization of fractional Hermite-Hadamard type integral inequalities which contain both Riemann integral and Riemann-Liouville fractional integral operators through convexity.

Keywords- Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, integral inequalities.

I. INTRODUCTION

Definition 1: The function $f:[a,b]\subset \mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[15, p 137], [9]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 9, 10, 15]) and the references cited therein.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [11, 12,14, 16].

Definition 2: Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\ge 0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Meanwhile, Sarikaya et al.[18] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha>0$.

Lemma 1: Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \quad (2)$$
$$=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)dt.$$

It is remarkable that Sarikaya et al.[18] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1: Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$
$$\le\frac{f(a)+f(b)}{2} \quad (3)$$

with $\alpha>0$.

Similar to (3) and (2), Sarikaya and Yildirim gave the following interesting results, in [20].

Theorem 2: Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)$$

$$\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right]$$

$$\le \frac{f(a)+f(b)}{2}$$

with $\alpha > 0.$

Lemma 2: Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b.$ If $f' \in L[a,b],$ then the following equality for fractional integrals holds:

$$\frac{f(a)+f(b)}{2}$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right]$$

$$=\frac{b-a}{4}\int_0^1 t^{\alpha}\begin{bmatrix} f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right) \\ -f'\left(\frac{t}{2}a+\frac{2-t}{2})b\right)\end{bmatrix}dt$$

For some recent results connected with fractional integral inequalities see ([3, 4, 5, 6, 7,8, 13, 17, 19, 19, 21]).

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for fractional integral operators using the generalized identity obtained for fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

II. MAIN RESULTS

Throughout this section, we denote $A=\left(\frac{m\alpha}{n}\right)^{\frac{1}{\alpha-1}}$ for $n,\alpha>0$ and $m \ge 0.$ We give two important fractional integral identities for differentiable functions as follow:

Lemma 3: Let $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b],$ then the following equality for fractional integrals holds:

$$\left(m-\frac{n}{\alpha}\right)\frac{f(a)+f(b)}{2}-\frac{m}{b-a}\int_a^b f(x)dx$$

$$+\frac{n\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right] \qquad (4)$$

$$=\frac{b-a}{2}\int_0^1\begin{bmatrix}(1-t)\left(m-\frac{n}{\alpha}(1-t)^{\alpha-1}\right) \\ -t\left(m-\frac{n}{\alpha}t^{\alpha-1}\right)\end{bmatrix}$$

$$\times f'(ta+(1-t)b)dt.$$

Proof: It suffices to note that

$$I=\int_0^1\left[(1-t)\left(m-\frac{n}{\alpha}(1-t)^{\alpha-1}\right)-t\left(m-\frac{n}{\alpha}t^{\alpha-1}\right)\right]$$

$$\times f'(ta+(1-t)b)dt$$

$$=\int_0^1(1-t)\left(m-\frac{n}{\alpha}(1-t)^{\alpha-1}\right)f'(ta+(1-t)b)dt$$

$$-\int_0^1 t\left[m-\frac{n}{\alpha}t^{\alpha-1}\right]f'(ta+(1-t)b)dt$$

$$=I_1-I_2$$

Integrating by parts, we obtain

$$I_1=\int_0^1(1-t)\left(m-\frac{n}{\alpha}(1-t)^{\alpha-1}\right)f'(ta+(1-t)b)dt$$

$$=\frac{1}{b-a}\left(m-\frac{n}{\alpha}\right)f(b)-\frac{m}{(b-a)^2}\int_a^b f(x)dx$$

$$+\frac{n\Gamma(\alpha)}{(b-a)^{\alpha+1}}J^{\alpha}_{b-}f(a),$$

and similarly we get

$$I_2=\int_0^1 t\left[m-\frac{n}{\alpha}t^{\alpha-1}\right]f'(ta+(1-t)b)dt$$

$$=-\frac{1}{b-a}\left(m-\frac{n}{\alpha}\right)f(a)+\frac{m}{(b-a)^2}\int_a^b f(x)dx$$

$$-\frac{n\Gamma(\alpha)}{(b-a)^{\alpha+1}}J^{\alpha}_{a+}f(b).$$

From I_1 and I_2 , it follows that

$$I=I_1-I_2$$

$$=\frac{1}{b-a}\left(m-\frac{n}{\alpha}\right)[f(a)+f(b)]$$

$$-\frac{2m}{(b-a)^2}\int_a^b f(x)dx \qquad (5)$$

$$+\frac{n\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]$$

If we multiply (5) by $\frac{b-a}{2}$, then we obtain the desired result.

Remark 1: If we choose $m=0$ and $n=\alpha$ in Lemma 3, then the identitiy (4) reduces to identity (2).

Lemma 4: Let $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b],$ then the following equality for fractional integrals holds:

$$\left(m-\frac{n}{\alpha}\right)f\left(\frac{a+b}{2}\right)-\frac{m}{b-a}\int_a^b f(x)dx$$

$$+\frac{n2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right] \qquad (6)$$

$$=\frac{b-a}{4}\int_0^1 t\left[m-\frac{n}{\alpha}t^{\alpha-1}\right]$$

$$\times\begin{bmatrix}f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right) \\ -f'\left(\frac{t}{2}a+\frac{2-t}{2})b\right)\end{bmatrix}dt.$$

Proof: Integrating by parts

$$I_3 = \int_0^1 t\left[m - \frac{n}{\alpha}t^{\alpha-1}\right] f'\left(\frac{t}{2}b + \frac{2-t}{2}a\right)dt$$

$$= \frac{2}{b-a}\left(m - \frac{n}{\alpha}\right) f\left(\frac{a+b}{2}\right) - m\left(\frac{2}{b-a}\right)^2 \int_a^{\frac{a+b}{2}} f(x)dx + n\Gamma(\alpha)\left(\frac{2}{b-a}\right)^{\alpha+1} J^{\alpha}_{\left(\frac{a+b}{2}\right)-} f(a),$$

and similarly we get

$$I_4 = \int_0^1 t\left[m - \frac{n}{\alpha}t^{\alpha-1}\right] f'\left(\frac{t}{2}b + \frac{2-t}{2}a\right)dt$$

$$= -\frac{2}{b-a}\left(m - \frac{n}{\alpha}\right) f\left(\frac{a+b}{2}\right) + m\left(\frac{2}{b-a}\right)^2 \int_{\frac{a+b}{2}}^b f(x)dx - n\Gamma(\alpha)\left(\frac{2}{b-a}\right)^{\alpha+1} J^{\alpha}_{\left(\frac{a+b}{2}\right)+} f(b).$$

Then, it follows that

$$\begin{aligned} &I_3 - I_4 \\ &= \frac{4}{b-a}\left(m - \frac{n}{\alpha}\right) f\left(\frac{a+b}{2}\right) - m\left(\frac{2}{b-a}\right)^2 \int_a^b f(x)dx \\ &+ n\Gamma(\alpha)\left(\frac{2}{b-a}\right)^{\alpha+1} \\ &\times \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)-} f(a)\right]. \end{aligned} \tag{7}$$

If we multiply (7) by $\frac{b-a}{4}$, then we obtain the required result.

Remark 2: If we choose $m = 0$ and $n = \alpha$ in Lemma 4, then Lemma 3 reduces to Lemma 2.

Theorem 3: Let $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\frac{n}{\alpha} \geq m$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:

$$\begin{aligned} &\left| \left(m - \frac{n}{\alpha}\right)\frac{f(a)+f(b)}{2} - \frac{m}{b-a}\int_a^b f(x)dx \right. \\ &\left. - \frac{n\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)\right] \right| \\ &\leq \frac{b-a}{2}\left[\frac{n}{\alpha(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right) - \frac{m}{4}\right] \\ &\times \left[|f'(b)| + |f'(a)|\right] \end{aligned} \tag{8}$$

Proof: From Lemma 3, we have

$$\begin{aligned} &\left| \left(m - \frac{n}{\alpha}\right)\frac{f(a)+f(b)}{2} - \frac{m}{b-a}\int_a^b f(x)dx \right. \\ &\left. - \frac{n\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)\right] \right| \\ &\leq \frac{b-a}{2}\int_0^1 \left| \begin{array}{l} (1-t)\left(m - \frac{n}{\alpha}(1-t)^{\alpha-1}\right) \\ -t\left(m - \frac{n}{\alpha}t^{\alpha-1}\right) \end{array} \right| \\ &\times |f'(ta + (1-t)b)|\,dt \\ &= \frac{b-a}{2}\left[\int_0^{\frac{1}{2}} \left[\begin{array}{l} t\left(m - \frac{n}{\alpha}t^{\alpha-1}\right) \\ -(1-t)\left(m - \frac{n}{\alpha}(1-t)^{\alpha-1}\right) \end{array} \right] \right. \\ &\times |f'(ta + (1-t)b)|\,dt \\ &\left. + \int_{\frac{1}{2}}^1 \left[\begin{array}{l} (1-t)\left(m - \frac{n}{\alpha}(1-t)^{\alpha-1}\right) \\ -t\left(m - \frac{n}{\alpha}t^{\alpha-1}\right) \end{array} \right] \right] \\ &\times |f'(ta + (1-t)b)|\,dt \end{aligned}$$

Since $|f'|$ is a mapping, we deduce that

$$\begin{aligned} &\int_0^{\frac{1}{2}} \left[t\left(m - \frac{n}{\alpha}t^{\alpha-1}\right) - (1-t)\left(m - \frac{n}{\alpha}(1-t)^{\alpha-1}\right)\right] \\ &\times |f'(ta + (1-t)b)|\,dt \\ &\leq |f'(a)|\left[\frac{n}{\alpha(\alpha+1)(\alpha+2)} - \frac{n}{\alpha(\alpha+1)2^{\alpha+1}} - \frac{m}{24}\right] \\ &+ |f'(b)|\left[\frac{n}{\alpha(\alpha+2)} - \frac{n}{\alpha(\alpha+1)2^{\alpha+1}} - \frac{5m}{24}\right] \end{aligned}$$

and similarly,

$$\begin{aligned} &\int_{\frac{1}{2}}^1 \left[(1-t)\left(m - \frac{n}{\alpha}(1-t)^{\alpha-1}\right) - t\left(m - \frac{n}{\alpha}t^{\alpha-1}\right)\right] \\ &\times |f'(ta + (1-t)b)|\,dt \\ &\leq |f'(a)|\left[\frac{n}{\alpha(\alpha+2)} - \frac{n}{\alpha(\alpha+1)2^{\alpha+1}} - \frac{5m}{24}\right] \\ &+ |f'(b)|\left[\frac{n}{\alpha(\alpha+1)(\alpha+2)} - \frac{n}{\alpha(\alpha+1)2^{\alpha+1}} - \frac{m}{24}\right]. \end{aligned}$$

which completes the proof.

Remark 3: If we choose $m = 0$ and $n = \alpha$ in Theorem 3, then Theorem 3 reduces to Theorem 3 proved by Sarikaya et al. in [18].

Theorem 4: Let $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex

on $[a,b]$, then the following inequality for fractional integrals holds:

$$\left|\left(m-\frac{n}{\alpha}\right)f\left(\frac{a+b}{2}\right)-\frac{m}{b-a}\int_a^b f(x)dx + \frac{n2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right]\right| \le \frac{b-a}{4}\left[\frac{m}{2}\left(2A^2-1\right)+\frac{n}{\alpha(\alpha+1)}\left(1-2A^{\alpha+1}\right)\right] \times\left[\left|f'(b)\right|+\left|f'(a)\right|\right].$$

Proof: From Lemma 4, using convexity of $|f'|$, we obtain

$$\left|\left(m-\frac{n}{\alpha}\right)f\left(\frac{a+b}{2}\right)-\frac{m}{b-a}\int_a^b f(x)dx + \frac{n2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right]\right| \le \frac{b-a}{4}\left[\frac{m}{2}\left(2A^2-1\right)+\frac{n}{\alpha(\alpha+1)}\left(1-2A^{\alpha+1}\right)\right] \times\left[\left|f'(b)\right|+\left|f'(a)\right|\right].$$

This completes the proof.

Remark 4: For $m=0$ and $n=\alpha>0$, we have the inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)_-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4(\alpha+1)}\left[\left|f'(b)\right|+\left|f'(a)\right|\right]$$

which was given by Sarikaya and Yildirim in [20].

REFERENCES

[1] A.G. Azpeitia, Convex functions and the Hadamard inequality, Rev. Colombiana Math., 28 (1994), 7-12.

[2] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.

[3] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.

[4] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Scinece, 9(4) (2010), 493-497.

[5] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.

[6] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.

[7] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

[8] J. Deng and J. Wang, Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. J. Inequal.Appl. 2013, Article ID 364 (2013)

[9] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[10] S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. lett., 11(5) (1998), 91-95.

[11] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Springer Verlag, Wien (1997), 223-276.

[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[13] M.A. Latif, S.S. Dragomir and A.E. Matouk, New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals, J. Fract. Calc. Appl. 2, 1-15 (2012)

[14] S. Miller and B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, USA, 1993, p.2.

[15] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.

[16] I. Podlubni, Fractional Differential Equations, Academic Press, San Diego, 1999.

[17] M.Z. Sarikaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, Abstract and Applied Analysis, Volume 2012 (2012), Article ID 428983, 10 pages.

[18] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, Hermite - Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048, 57 (2013) 2403--2407.

[19] M.Z.Sarikaya, H. Filiz and M.E. Kiris, On some generalized integral inequalities for Riemann-Liouville fractional integrals, Filomat, 29:6 (2015), 1307--1314.

[20] M.Z. Sarikaya and H. Yildirim, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals,Miskolc Mathematical Notes, Vol. 17 (2016), No. 2, pp. 1049--1059.

[21] Y. Zhang and J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, J. Inequal. Appl. 2013, Article ID 220 (2013)

★★★