SOME NEW CHEBYSHEV TYPE INTEGRAL INEQUALITIES VIA FRACTIONAL INTEGRAL OPERATOR WITH EXPONENTIAL KERNEL

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Abstract- In this paper, we provide some new Chebyshev type integral inequalities by using fractional integral operators with exponential kernel for two synchronous functions on $[0,\infty)$, which means that they are having the same sense of variation on $[0,\infty)$.

Keywords- Chebyshev inequality, fractional integral operators with kernel.

I. INTRODUCTION

In 1882, Chebyshev [3] proved the following inequality:

Theorem 1: Let f and g be two integrable functions in $[0,1]$ .If both functions are simultaneously increasing or decreasing for the same values of x in $[0,1]$, then

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx \qquad (1)$$

If one function is increasing and the other is decreasing for the same values of x in $[0,1]$, then (1) reverses.

Also, author gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \qquad (2)$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \qquad (3)$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$. In the last years, many papers were devoted to the generalization of the inequalities (1) and (2), we can mention the works [1],[2],[4]-[7],[9]-[14].

In addition to this, recently, Kirane and Torebek [8] have introduce a new class of fractional integrals which are summarised as follow:

Definition 1: Let $f \in L_1(a,b)$. The fractional integrals $\mathbf{I}_{a+}^{\alpha}$ and $\mathbf{I}_{b-}^{\alpha}$ of order $\alpha \in (0,1)$ are defined by

$$\mathbf{I}_{a+}^{\alpha}(f)(x) = \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)dt,$$

$x > a$

and

$$\mathbf{I}_{b-}^{\alpha}(f)(x) = \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)dt,$$

$x < b.$

II. MAIN RESULTS

In this section, we will present some fractional integral inequalities for functions defined on the positive real line with the help of fractional integral operators with exponential kernel given above. The inequalities to be given in this section are a generalization of the well-known Chebyshev inequalities.

Theorem 2: Let f and g be two synchronous functions on $[0,\infty)$. Then for all $x > 0$ we have

$$\mathbf{I}_{0+}^{\alpha}(1)(x)\mathbf{I}_{0+}^{\alpha}(fg)(x) \geq \mathbf{I}_{0+}^{\alpha} f(x)\mathbf{I}_{0+}^{\alpha} g(x) \qquad (4)$$

where $\alpha \in (0,1)$.

Proof: Let f and g be two functions satisfying the conditions of Theorem t1. Then, we have

$$(f(\eta)-f(\xi))(g(\eta)-g(\xi)) \geq 0.$$

Therefore

$$f(\eta)g(\eta) + f(\xi)g(\xi) \geq f(\eta)g(\xi) + f(\xi)g(\eta). \qquad (5)$$

Then by multiplying inequality (5) by $\frac{1}{\alpha^2}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\eta)\right\}$, we find that

$$\begin{aligned}
&\frac{1}{\alpha^2}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\alpha}{\alpha}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\eta)+f(\xi)g(\xi)\right]\\
&\geq\frac{1}{\alpha^2}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\alpha}{\alpha}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\xi)+f(\xi)g(\eta)\right]
\end{aligned}\tag{6}$$

By double integrating (6) with respect to η and ξ over $(0,x)\times(0,x)$, we deduce that

$$\begin{aligned}
&\frac{1}{\alpha^2}\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\alpha}{\alpha}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\eta)+f(\xi)g(\xi)\right]d\xi d\eta\\
&\geq\frac{1}{\alpha^2}\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\alpha}{\alpha}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\xi)+f(\xi)g(\eta)\right]d\xi d\eta.
\end{aligned}$$

We get the desired result after rearranging the above inequality.

Theorem 3: Let f and g be two synchronous functions on $[0,\infty)$, that is they are having the same sense of variation on $[0,\infty)$. Then for all $x>0$ we have

$$\begin{aligned}
&\mathbf{I}^{\beta}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(fg)(x)+\mathbf{I}^{\alpha}_{0+}(1)(t)\mathbf{I}^{\beta}_{0+}(fg)(x)\\
&\geq\mathbf{I}^{\alpha}_{0+}f(x)\mathbf{I}^{\beta}_{0+}g(x)+\mathbf{I}^{\beta}_{0+}f(x)\mathbf{I}^{\alpha}_{0+}g(x)
\end{aligned}$$

where $\alpha,\beta\in(0,1)$.

Proof: First of all, if we multiply (5) by $\frac{1}{\alpha\beta}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha\beta}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\eta)+f(\xi)g(\xi)\right]\\
&\geq\frac{1}{\alpha\beta}\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\xi)+f(\xi)g(\eta)\right]
\end{aligned}\tag{7}$$

By double integrating (7) with respect to η and ξ over $(0,x)\times(0,x)$, we deduce that

$$\begin{aligned}
&\frac{1}{\alpha\beta}\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\eta)+f(\xi)g(\xi)\right]d\xi d\eta\\
&\geq\frac{1}{\alpha\beta}\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\\
&\times\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\\
&\times\left[f(\eta)g(\xi)+f(\xi)g(\eta)\right]d\xi d\eta.
\end{aligned}$$

The proof is completed by a simple calculation.

Theorem 4: Let $(f_i)_{i=1,2,\ldots,n}$ be n positive increasing functions on $[0,\infty)$. Then for all $x>0$ we have

$$\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{n-1}\mathbf{I}^{\alpha}_{0+}\left(\prod_{i=1}^{n}f_i\right)(x)\geq\prod_{i=1}^{n}\left(\mathbf{I}^{\alpha}_{0+}f_i(x)\right)$$

where $\alpha\in(0,1)$.

Proof: We prove this theorem by Mathematical induction. Clearly, for $n=1$, we have $\mathbf{I}^{\alpha}_{0+}f_1(x)\geq\mathbf{I}^{\alpha}_{0+}f_1(x)$.

For $n=2$, applying Theorem t1, we obtain

$$\mathbf{I}^{\alpha}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(f_1f_2)(x)\geq\mathbf{I}^{\alpha}_{0+}f_1(x)\mathbf{I}^{\alpha}_{0+}f_2(x).$$

Now suppose that

$$\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{k-1}\mathbf{I}^{\alpha}_{0+}\left(\prod_{i=1}^{k}f_i\right)(x)\geq\prod_{i=1}^{k}\left(\mathbf{I}^{\alpha}_{0+}f_i(x)\right)$$

for $k\in\overline{3,n-1}$.

Because $(f_i)_{i=1,2,\ldots,n}$ be n positive increasing functions on $[0,\infty)$, then $\left(\prod_{i=1}^{k}f_i\right)(t)$ is an increasing function. Thus we can apply the result of Theorem 2 to the functions $g=\prod_{i=1}^{k}f_i$ and $f=f_{k+1}$. We have:

$$\begin{aligned}
&\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{k}\mathbf{I}^{\alpha}_{0+}\left(\prod_{i=1}^{k+1}f_i\right)(x)\\
&=\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{k-1}\mathbf{I}^{\alpha}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(g.f)(x)\\
&\geq\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{k-1}\mathbf{I}^{\alpha}_{0+}g(x)\mathbf{I}^{\alpha}_{0+}f(x)\\
&=\left(\mathbf{I}^{\alpha}_{0+}(1)(x)\right)^{k-1}\mathbf{I}^{\alpha}_{0+}\left(\prod_{i=1}^{k}f_i\right)(x)\mathbf{I}^{\alpha}_{0+}f_{k+1}(x)\\
&\geq\prod_{i=1}^{k}\left(\mathbf{I}^{\alpha}_{0+}f_i(t)\right)\mathbf{I}^{\alpha}_{0+}f_{k+1}(t)\\
&=\prod_{i=1}^{k+1}\left(\mathbf{I}^{\alpha}_{0+}f_i(x)\right)
\end{aligned}$$

which completes the proof.

Theorem 5: Let f and g be two differentiable function on $[0,\infty)$. Then for all $t,\rho>0$, $\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have

$$\left|\mathbf{I}^{\beta}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(fg)(x)+\mathbf{I}^{\alpha}_{0+}(1)(t)\mathbf{I}^{\beta}_{0+}(fg)(x)-\mathbf{I}^{\alpha}_{0+}f(x)\mathbf{I}^{\beta}_{0+}g(x)-\mathbf{I}^{\beta}_{0+}f(x)\mathbf{I}^{\alpha}_{0+}g(x)\right|\le\frac{\|f'\|_{\infty}\|g'\|_{\infty}}{\alpha\beta}x^{2}\mathbf{I}^{\alpha}_{0+}(1)(x)\mathbf{I}^{\beta}_{0+}(1)(x)$$

where

$$\|f'\|_{\infty}=\sup_{x\in[0,\infty)}|f'(x)|<\infty,$$

and

$$\mathbf{I}^{\gamma}_{0+}(1)(x)=\frac{\gamma}{1-\gamma}\left[1-\exp\left\{-\frac{1-\gamma}{\gamma}x\right\}\right].$$

Proof: With basic calculation, we have

$$\mathbf{I}^{\beta}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(fg)(x)+\mathbf{I}^{\alpha}_{0+}(1)(t)\mathbf{I}^{\beta}_{0+}(fg)(x)$$

$$-\mathbf{I}^{\alpha}_{0+}f(x)\mathbf{I}^{\beta}_{0+}g(x)-\mathbf{I}^{\beta}_{0+}f(x)\mathbf{I}^{\alpha}_{0+}g(x)$$

$$=\frac{1}{\alpha\beta}\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\times(f(\eta)-f(\xi))(g(\eta)-g(\xi))d\eta d\xi.$$

Then taking modulus of the above equality, we find that

$$\left|\mathbf{I}^{\beta}_{0+}(1)(x)\mathbf{I}^{\alpha}_{0+}(fg)(x)+\mathbf{I}^{\alpha}_{0+}(1)(t)\mathbf{I}^{\beta}_{0+}(fg)(x)-\mathbf{I}^{\alpha}_{0+}f(x)\mathbf{I}^{\beta}_{0+}g(x)-\mathbf{I}^{\beta}_{0+}f(x)\mathbf{I}^{\alpha}_{0+}g(x)\right|$$

$$=\frac{1}{\alpha\beta}\left|\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\times(f(\eta)-f(\xi))(g(\eta)-g(\xi))d\eta d\xi\right|$$

$$=\frac{1}{\alpha\beta}\left|\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}\times\left(\int_{\xi}^{\eta}f'(u)du\right)\left(\int_{\xi}^{\eta}g'(v)dv\right)d\eta d\xi\right|$$

$$\le\frac{\|f'\|_{\infty}\|g'\|_{\infty}}{\alpha\beta}\times\left|\int_0^x\int_0^x\exp\left\{-\frac{1-\alpha}{\alpha}(x-\xi)\right\}\times\exp\left\{-\frac{1-\beta}{\beta}(x-\eta)\right\}(\eta-\xi)^2d\eta d\xi\right|$$

$$\le\frac{\|f'\|_{\infty}\|g'\|_{\infty}}{\alpha\beta}x^{2}\mathbf{I}^{\alpha}_{0+}(1)(x)\mathbf{I}^{\beta}_{0+}(1)(x)$$

which completes the proof.

CONCLUDING REMARKS

In this study, some new Chebyshev type integral inequalities for two synchronous functions are presented and proposed and proved with the help fractional integral inequalities with exponential kernel. The established results are generalization of some existing Chebyshev type integral inequalities in the previous published studies.

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