A METHOD FOR SOLVING OPTIMAL CONTROL PROBLEMS SUBJECT TO PROBABILISTIC AFFINE STATE CONSTRAINTS FOR LINEAR DISCRETE-TIME UNCERTAIN SYSTEMS

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Abstract — Recently, optimal control problems subject to probabilistic constraints have attracted much attention in many research fields. Although it is not straightforward to handle probabilistic constraints in an optimization problem, several methods haven been proposed to handle probabilistic constraints. This paper examines probabilistic constrained optimal control problems for linear discrete-time systems under uncertain disturbances with unknown probability distributions. The objective of this paper is to provide a method for solving the optimal control problems subject to probabilistic affine state constraints.

Index Terms — Optimal Control, Stochastic Systems, Discrete-Time Systems, Probabilistic Constraints.

I. INTRODUCTION

So far, optimal control problems subject to deterministic constraints for linear discrete-time systems under uncertain disturbances have been solved in [1]--[14]. Recently, probabilistic constrained optimal control problems have attracted much attention. Probabilistic constraints are dealt with by stochastic optimal control problems where expected values of performance indices, probabilistic constraints and convergence in probability are considered by exploiting the statistical information on the system parameters [15]--[27].

However, probabilistic constraints are not directly tractable in an optimization problem. In recent decades, much attention has been paid to this difficulty of the stochastic optimal control problem. Thus, several methods [15]--[21] haven been proposed to handle probabilistic constraints. Although the papers [15]--[21] have achieved significant progress in dealing with probabilistic constraints of the stochastic optimization problems, there are several restrictions imposed on the probability distributions of stochastic disturbances such as the normal (Gaussian) distribution, known distribution, finite support, and time invariance. In contrast, we address here unknown arbitrary probability distributions including non-Gaussian, infinitely supported, and time-variant distributions.

The sampling methods using scenario approximation [22]--[24] are alternative methods for arbitrary probability distributions. However, the disadvantage of sampling methods is its heavy computational load. In [25], an ellipsoid approximation method was proposed to handle probabilistic constraints. However, the calculation of the maximum volume inscribed ellipsoid also requires computational load. In [26]--[29], direct component-wise comparison methods using the multi-dimensional Chebyshev's inequality was proposed to address probabilistic constraints without using ellipsoid approximation. However, the tractable constraints in [26]--[29] are restricted to the component-wise state constraints. That means affine state constraints cannot be addressed by the methods in [26]--[29]. Hence, we consider here affine state constraints in stochastic optimization problems.

In this paper, we address probabilistic affine state constraints in optimal control problems for linear discrete-time systems under uncertain disturbances with unknown probability distributions. The objective of this paper is to provide a method for solving the optimal control problems subject to probabilistic affine state constraints.

II. NOTATION AND SYSTEM MODEL

Notation used in this paper is same as the one introduced in [27]. Throughout this paper, we consider the following linear discrete-time system with stochastic disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Cw(t) \tag{1}$$

where t is the time step, x is the state, u is the control input, and w is the unknown stochastic disturbance.The system coefficients A , B , and C are all known as constant matrices. The pair (A, B) is assumed to be controllable. We also assume that the initial state $x(0)$ is given and that all components of state $x(t)$ are deterministically observable.

Next, we introduce some assumptions about the properties of the stochastic disturbances.

Assumption 1. Random variables $w_i(t)$ and $w_j(t)$ are mutually independent for all i, j, t.

In fact, most previous studies typically assumed that random variables are mutually independent as well as

Assumption 1. The case where random variables are mutually correlated requires more complicated analysis than the one provided here because the covariance of w cannot be neglected.

Assumption 2. $E\left(w(t)\right)$ and $V\left(w(t)\right)$ are assumed to be known for each time.

Note that the probability distributions of random variables w are not necessarily assumed to be known. However, the probability distributions were assumed to be known in previous studies [15]-[21] to transform the probabilistic constraints into deterministic constraints. In the present study, the assumption related to known probability distributions is relaxed to include arbitrary unknown probability distributions.

II. PROBLEM STATEMENT

Hereafter, we formulate the stochastic optimal control problem of system (1). The control input at each time is determined to minimize the performance index given by

$$J := \phi\left[x(t+N)\right] + \sum_{k=t}^{t+N+1} L\left[x(k),u(k)\right]. \tag{2}$$

Therein, N denotes the length of the prediction horizon. ϕ and L are defined by

$$\phi := E\left[x'(t+N)Px(t+N)\right],$$

$$L := E\left[x'(k)Qx(k)\right] + u'(k)Ru(k),$$

where P, Q and R are positive definite constant matrices. ϕ is the terminal cost function, and L is the stage cost function over the prediction horizon.

Let

$$p(t) = \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix} \in \begin{bmatrix} 0 & 1 \end{bmatrix}^n$$

denote the probability in vector form, which means that each component belongs to $\begin{bmatrix} 0 & 1 \end{bmatrix}$ for each time.

For notational convenience, we introduce the following matrices:

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix},$$

$$\mathbf{U}(t) := \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix},$$

$$\mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C & 0 & \cdots & 0 \\ AC & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}C & A^{N-2}C & \cdots & C \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix},$$

$$\mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix},$$

$$\mathbf{p}(t) = \begin{bmatrix} p(t+1) \\ \vdots \\ p(t+N) \end{bmatrix}.$$

Using the aforementioned notation, we rewrite the performance index in (2) as follows:

$$J[x(t),\mathbf{X}(t),\mathbf{U}(t)] = E\left[x'(t)Qx(t)\right] + E\left[\mathbf{X}'(t)\mathbf{Q}\mathbf{X}(t)\right] + \mathbf{U}'(t)\mathbf{R}\mathbf{U}(t). \tag{3}$$

In addition, (1) over the prediction horizon can be rewritten as

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathbf{W}(t). \tag{4}$$

In [26]--[29], we considered the probabilistic component-wise state constraints shown below. Let $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of x, respectively.We imposed the following probabilistic constraint on the optimization problem:

$$P\left(\underline{x}_i(k) \le x_i(k) \le \overline{x}_i(k)\right) \ge p_i(k) \tag{5}$$

where $\underline{x}_i, \overline{x}_i$ and p_i are given constant sequences and their subscript indicates the i-th element of the vector.Condition (5) indicates that state x_i over the prediction horizon must remain within the bound $\begin{bmatrix} \underline{x}_i & \overline{x}_i \end{bmatrix}$ at least with probability p_i.Let $\underline{\mathbf{X}}$ and $\overline{\mathbf{X}}$ be defined by

$$\underline{\mathbf{X}}(t) := \begin{bmatrix} \underline{x}(t+1) \\ \vdots \\ \underline{x}(t+N) \end{bmatrix},$$

$$\overline{\mathbf{X}}(t) := \begin{bmatrix} \overline{x}(t+1) \\ \vdots \\ \overline{x}(t+N) \end{bmatrix}.$$

Using the above notation, probabilistic constraint (5) is rewritten in vector form as

$$P\left(\underline{\mathbf{X}}(t) \le \mathbf{X}(t) \le \overline{\mathbf{X}}(t)\right) \ge \mathbf{p}(t) \tag{6}$$

More precisely, by using the components of the vectors, condition (6) can be described as

$$\bigwedge_{i=1}^{nN} P\left(\underline{\mathbf{X}}_i(t) \le \mathbf{X}_i(t) \le \overline{\mathbf{X}}_i(t)\right) \ge \mathbf{p}_i(t) \tag{7}$$

where notation $\wedge$ denotes the logical conjunction.

In this study, the probabilistic component-wise state constraints (6) are generalized to the probabilistic affine state constraints shown below.

$$P\left(\mathbf{D}\mathbf{X}(t) \le \mathbf{h}\right) \ge \mathbf{p}(t). \tag{8}$$

where **D** and **h** are given constant parameters.

IV. SOLUTION OF STOCHASTIC OPTIMAL CONTROL PROBLEM

In this section, we provide the solution of stochastic optimal control problem. First, we transform the minimization problem of (2) subject to (1) into a quadratic programming problem with respect to the sequence of control inputs over the prediction horizon.

From (4), $E\left(\mathbf{X}(t)\right)$ and $V\left(\mathbf{X}(t)\right)$ are given by

$$E(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}E(\mathbf{W}(t)), \tag{9a}$$

$$V(\mathbf{X}(t)) = \mathbf{C} \otimes \mathbf{C}V(\mathbf{W}(t)). \tag{9b}$$

Moreover, (3) indicates that

$$\begin{aligned} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = {} & x'(t)Qx(t) + \mathbf{U}'(t)\mathbf{R}\mathbf{U}(t) \\ & tr\left[QC_v(\mathbf{X}(t))\right] + E(\mathbf{X}(t))'\mathbf{Q}E(\mathbf{X}(t)) \end{aligned} \tag{10}$$

Note that covariance matrix $C_v(\mathbf{X}(t))$ is independent of $\mathbf{U}$.

Substituting (9a) into (10) and neglecting the terms that do not contain $\mathbf{U}$, we obtain

$$\min_{\mathbf{U}(t)} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \min_{\mathbf{U}(t)} \left\{ \begin{array}{l} \mathbf{U}'(t)\left(\mathbf{B}'\mathbf{Q}\mathbf{B} + \mathbf{R}\right)\mathbf{U}(t) \\ +2\left(\mathbf{A}x(t) + \mathbf{C}E(\mathbf{W}(t))\right)'\mathbf{Q}\mathbf{B}\mathbf{U}(t) \end{array} \right\} \tag{11}$$

Note that the minimization problem of J in (2) subject to (1) has been reduced to a quadratic programming problem with respect to $\mathbf{U}$.

In general, however, solving the quadratic programming problem with probabilistic constraints is not straightforward. Below, we provide the method for solving stochastic optimal control problems.

The inequality shown below is known as the Cantelli's inequality.

Lemma 1. ([31]) For any random variable x and positive constant $\kappa > 0$, the following inequality holds:

$$P\left(x - E(x) \ge \kappa\right) \le \frac{V(x)}{V(x) + \kappa^2} \tag{12}$$

We convert the probabilistic constraint into deterministic constraint using the inequality in Lemma 1.

Theorem 1. Suppose that the following condition holds:

$$\mathbf{D}\mathbf{B}\mathbf{U}(t) \le \mathbf{h} - \mathbf{D}\left(\mathbf{A}x(t) + \mathbf{C}E(\mathbf{W}(t))\right) - \mathbf{V}(t), \tag{13}$$

Where the *i*-th element of $\mathbf{V}$ is given by

$$\mathbf{V}_i = \sqrt{\frac{\mathbf{p}_i}{1 - \mathbf{p}_i}\left(\mathbf{D} \otimes \mathbf{D}(\mathbf{C} \otimes \mathbf{C})V(\mathbf{W}(t))\right)}. \tag{14}$$

Then, the probabilistic condition (8) is fulfilled.

Proof. Using Lemma 1, we have the following inequality in the component-wise form:

$$P\left((\mathbf{D}\mathbf{X})_i - E(\mathbf{D}\mathbf{X})_i \ge \kappa\right) \le \frac{V(\mathbf{D}\mathbf{X})_i}{V(\mathbf{D}\mathbf{X})_i + \kappa^2}, \tag{15}$$

where subscript *i* denotes the *i*-th element of a vector. Accordingly, we have the following inequality:

$$P\left((\mathbf{D}\mathbf{X})_i - E(\mathbf{D}\mathbf{X})_i < \kappa\right) \ge 1 - \frac{V(\mathbf{D}\mathbf{X})_i}{V(\mathbf{D}\mathbf{X})_i + \kappa^2}, \tag{16}$$

Suppose that the following equation holds:

$$\mathbf{p}_i = 1 - \frac{V(\mathbf{D}\mathbf{X})_i}{V(\mathbf{D}\mathbf{X})_i + \kappa^2}. \tag{17}$$

Then, it follows from (17) that

$$\kappa = \mathbf{V}_i. \tag{18}$$

Consequently, we have the following inequality:

$$P\left((\mathbf{D}\mathbf{X})_i - E(\mathbf{D}\mathbf{X})_i < \mathbf{V}_i\right) \ge \mathbf{p}_i. \tag{19}$$

For notational simplicity, we rewrite inequality (19) in a vector form, i.e.,

$$P\left(\mathbf{D}\mathbf{X} - E(\mathbf{D}\mathbf{X}) < \mathbf{V}\right) \ge \mathbf{p}. \tag{20}$$

Note that if the following condition

$$E(\mathbf{D}\mathbf{X}) + \mathbf{V} \le \mathbf{h} \tag{21}$$

is satisfied, then the probabilistic condition (8) is fulfilled. Substituting (9a) into (21), we can see that the condition (21) is equivalent to condition (13). Therefore, we conclude that if the deterministic constraint (13) on $\mathbf{U}$ is satisfied, then the probabilistic constraint (8) on $\mathbf{X}$ is also satisfied. This completes the proof.

Remark 1. From Theorem 1, the minimization problem of (11) with probabilistic constraint (6) is reduced to a quadratic programming problem with deterministic constraint (21), which can be solved using a conventional algorithm [32].

Remark 2. Consider an example of affine state constraint (8) as shown below:

$$P\left(\sum_{i=1}^{nN} \mathbf{X}_i \le h\right) \ge p. \tag{22}$$

In contrast, the following condition is an example of component-wise state constraint (6).

$$P\left(\mathbf{X} \le \overline{\mathbf{X}}\right) \ge \mathbf{p}. \tag{23}$$

Suppose that $\overline{\mathbf{X}}_i$ is given by $\overline{\mathbf{X}}_i = h / nN$ for all $i = 1, \cdots, nN$. Then, we have the following condition:

$$P\left(\sum_{i=1}^{nN} \mathbf{X}_i \le h\right) \ge \beta, \tag{24}$$

$$\beta := \prod_{i=1}^{nN} \mathbf{p}_i.$$

Comparing (22) with (24), we can see that p_i in (24) takes an underestimated value compared with p_i in (22). This is the advantage of applying probabilistic constraint (8) rather than (6).

CONCLUSION

In this paper, the optimal control problems subject to probabilistic constraints were investigated for linear discrete-time systems under uncertain disturbances with unknown probability distributions. Cantelli's inequality was applied to successfully handle probabilistic affine state constraints with a low computational load. Thus, the optimal control problems with probabilistic constraints were reduced to quadratic programming problems with deterministic constraints that are solved using a conventional algorithm. The development of an output-feedback-based method and a method to consider mixed state-input constraints based on the proposed method are possible future research areas.

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