MULTISTAGE BERNSTEIN POLYNOMIALS FOR SOLVING STIFF SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract- This paper aims to apply a new modification of the Bernstein polynomials method called Multistage Bernstein polynomials method (MB-polynomials) to solve stiff systems of ordinary differential equations. The MB-polynomials is a simple reliable modification based on the adaptation of the standard Bernstein polynomials method. The results of the MB-polynomials are compared with traditional Bernstein polynomials method and several other methods that solve stiff system of ordinary differential equations. The results revealed the accuracy of the proposed method.

I. INTRODUCTION

Stiff problems have been studied in many areas of chemical engineering, non linear mechanics, biochemistry and life sciences. Since 30 years ago, there have been many respected work focusing on the development of more advanced and efficient methods to solve stiff problems. BDF is formula that has been based on backward differentiation, and many modifications have been introduced by different authors to extend backward differentiation. The first modification of the formula is (EBDF) formula introduces by [6]. A-EBDF is also a modification of (BDF) and was applied to also solve stiff system of ordinary differential equations [9]. Homotopy perturbation methods (HPM) are well addressed in [8] are used for solving functional equations. Modification of the Homotopy perturbation methods which is called Rational Homotopy perturbation method (RHPM) is used likewise to obtain an analytic approximation of stiff system of ordinary differential equations [5]. However one of the most important analytic method for solving linear and nonlinear equations is Bernstein polynomials (B-polynomials) which is well addressed in [4, 1, 11, 3, 10, 2].

This paper presents a new modification of Bernstein polynomials which is called multistage Bernstein polynomials MB-polynomials, and it will overcome the error found in the results of stiff system which can be solved by the standard Bernstein polynomials method. The solution achieved by MB-polynomials is valid in larger x values than the standard B-polynomials. The results of MB-polynomials are compared with B-polynomials, BDF, EBDF methods, and A-EBDF, to illustrate that MB-polynomials are able to obtains more accurate results. This research paper is organized as follows: In Section 2, some basic definitions of the stiff system of ordinary differential equations are presented. Section 3 is devoted to describe the standard B-polynomials and the MB-polynomials. Section 4 illustrates that numerical comparisons with several other methods prove that the MB-polynomials method is a simple, yet powerful that provides method to give the approximate solutions for stiff system of ordinary differential equations. The final section presents the conclusions and recommendations for future work and the paper is deduced in Section 5 with a presentation of various deductions and recommendations for future work.

II. STIFF SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Consider a stiff system of ODEs:

$$\frac{dy_j}{dx} + f_j(x, y_1, y_2, \ldots, y_n) = g_j(x), \tag{1}$$

subject to the initial conditions

$$y_j(x_0) = \alpha_i, \tag{2}$$

where αj are constants, (j = 1, 2, . . . n).
First, the system is written (1) in the form of

$$\frac{dy_j}{dx} + f_j(x, y_1, y_2, \ldots, y_n) - g_j(x) = 0, \tag{3}$$

subject to the initial conditions (2). Presented below are the approaches to find the solution for (1) based on the standard B-polynomials and MB-polynomials respectively.

3 Solution by Bernstein polynomials (B-polynomials)
The Bernstein polynomials of degree m are defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

There are m+1 nth-degree Bernstein polynomials. For mathematical convenience, the equation Bi;m = 0, if i < 0 or i > m is often used.

In general, any function yj(x) with the first (m + 1) Bernstein polynomials is approximated as

$$y_j(x) = \sum_{i=0}^{m} c_{j,i} B_{i,m}(x) = \mathrm{C}_j^{\mathrm{T}} \phi(x), \quad j = 1, 2 \ldots, n, \tag{4}$$

where

$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^{\mathrm{T}}.$$

For $j = 1$,

$$y_1(x) = \sum_{i=0}^{m} c_{1,i} B_{i,m}(x) = \mathbf{C}_1^{\mathrm{T}} \phi(x),$$

where

$$\mathbf{C}_1^{\mathrm{T}} = [c_{1,0}, c_{1,1}, \ldots, c_{1,m}].$$

For $j = 2$,

$$y_2(x) = \sum_{i=0}^{m} c_{2,i} B_{i,m}(x) = \mathbf{C}_2^{\mathrm{T}} \phi(x),$$

where

$$\mathbf{C}_2^{\mathrm{T}} = [c_{2,0}, c_{2,1}, \ldots, c_{2,m}].$$

For the general case $j = n$,

$$y_n(x) = \sum_{i=0}^{m} c_{m,i} B_{i,m}(x) = \mathbf{C}_n^{\mathrm{T}} \phi(x),$$

where

$$\mathbf{C}_n^{\mathrm{T}} = [c_{n,0}, c_{n,1}, \ldots, c_{n,m}].$$

The derivatives of the vector $\phi(x)$ can be expressed as

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \mathbf{D}^{\mathbf{1}} \phi(x),$$

where D^1 is the $(m+1) \times (m+1)$ operational matrix of derivative given in [4, 1]. By means of the operational matrix of derivative [11], we approximate $g(x)$ as

$$g_j(x) \simeq \mathrm{G}_j^{\mathrm{T}} \phi(x), \quad j = 1, \ldots, n, \tag{6}$$

where the vector $\mathrm{G}_j^{\mathrm{T}} = [g_{j,0}(x), \ldots, g_{j,m}(x)]^{\mathrm{T}}$, by applying (4)–(6) on system (3), the residual $\Re(x)$ is obtained as

$$\begin{aligned}
\mathrm{C}_1^{\mathrm{T}} D^1 \phi(x) + f_1[x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)] - \mathrm{G}_1^{\mathrm{T}} \phi(x) &= 0, \\
\mathrm{C}_2^{\mathrm{T}} D^1 \phi(x) + f_2[x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)] - \mathrm{G}_2^{\mathrm{T}} \phi(x) &= 0, \\
&\vdots \\
\mathrm{C}_n^{\mathrm{T}} D^1 \phi(x) + f_n[x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)] - \mathrm{G}_n^{\mathrm{T}} \phi(x) &= 0,
\end{aligned} \tag{7}$$

with the initial conditions

$$C_1^T\phi(x_0) = \alpha_1, \quad C_2^T\phi(x_0) = \alpha_2, \quad \ldots, \quad C_n^T\phi(x_0) = \alpha_n. \tag{8}$$

To solve system (7), collocation points $x_0, x_1, \ldots, x_m$ need to be established which will be substituted in (7), then $m-1$ equations are obtained, with the initial conditions in (8). Now we have m equations where the unknowns are c_i, which can be solved by using Newton's iterative method. To find the collocation points $x_0, x_1, \ldots, x_m$, proceed as follows

$$\begin{pmatrix} B_{0,m}(x) - B_{0,m-1}(x) \\ B_{1,m}(x) - B_{1,m-1}(x) \\ \vdots \\ B_{m-1,m}(x) - B_{m-1,m-1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{9}$$

The points which are presented by (9) are called the collocation points $(x_0, x_1, \ldots, x_m)$. Therefore, according to B-polynomials, the m-term approximations for the solutions of (1) can be expressed as

$$y_1(x) = C_1^T\phi(x), \quad y_2(x) = C_2^T\phi(x), \quad \ldots, \quad y_n(x) = C_n^T\phi(x). \tag{10}$$

III. SOLUTİON BY MULTİSTAGE BERNSTEİN POLYNOMİALS (MB-POLYNOMİALS)

The approximate solution (10) is generally, as will be shown later, not valid for large x. For this reason, this method (MB-polynomials) is presented below, which is a simple way of ensuring validity of the approximation for large x to treat (10) as an algorithm for approximating the solutions of (1) in a sequence of intervals.
The initial approximation is chosen as

$$\begin{aligned} y_{1,0}(x) &= y_1(x^*) = \alpha_1^*, \\ y_{2,0}(x) &= y_2(x^*) = \alpha_2^*, \\ &\vdots \\ y_{n,0}(x) &= y_n(x^*) = \alpha_n^*, \end{aligned} \tag{11}$$

where x∗ is the left end point of each interval. Now, the unknowns yj;k(x), (j = 1, 2, . . . , n, k = 0, 1, . . .), are solved by applying the collocation method in (9). In order to carry out the iteration in every subinterval of equal length Δx, [0, x1), [x1, x2), [x2, x3) . . . [xk−1, x), it is necessary to know the values of the following:

$$y_{1,0}^*(x) = y_1(x^*), \quad y_{2,0}^*(x) = y_2(x^*), \quad \ldots, \quad y_{n,0}^*(x) = y_n(x^*). \tag{12}$$

A simple way for obtaining the necessary value (12) could be conducted by means of the previous $m-$term approximation of the preceding subinterval given by (10), that is

$$y_1^*(x) = C_1^T\phi(x^*), \quad y_2^*(x) = C_2^T\phi(x^*), \quad \ldots, \quad y_n^*(x) = C_n^T\phi(x^*). \tag{13}$$

To get more accurate and efficient results, attention must be paid to collocation points for each subinterval; so that it is possible to subdivide the interval and arrange the stages of the method as follows:

- Stage 1. Apply the method on the intervals $[0, x_1), [x_1, x_2), [x_2, x_3), \ldots, [x_{k-1}, 1)$, the collocation points $(x_0, x_1, \ldots, x_m)$ which will be used are the points presented from (9).
- Stage 2. Apply the method on the intervals $[1, x_1), [x_1, x_2), [x_2, x_3), \ldots, [x_{k-1}, 2)$, but the collocation points which will be used are $(x_0+1, x_1+1, \ldots, x_m+1)$.
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- Stage r. Apply the method on the intervals $[r-1, x_1), [x_1, x_2), [x_2, x_3), \ldots, [x_{k-1}, r)$, where $r = 1, 2, \ldots$, and the collocation points which will be used are $(x_0+(r-1), x_1+(r-1), \ldots, x_m+(r-1))$.

IV. NUMERİCAL EXPERİMENTS

To illustrate the effectiveness of the presented method, the following examples of stiff systems of ODEs have been provided.

Example 1.

Consider the stiff system of ODE [5],

$$\begin{cases} y_1'(x) = -1000y_1(x), \\ y_2'(x) = 0.909y_1(x) - y_2(x), \end{cases}$$

with the initial conditions $y_1(0) = 1, y_2(0) = 0.999$. The exact solution is

$$\begin{cases} y_1(x) = \mathrm{e}^{-1000x}, \\ y_2(x) = -\frac{0.909}{999}\mathrm{e}^{-1000x} + \frac{998.91}{999}\mathrm{e}^{-x}. \end{cases}$$

The B-polynomials and the MB-polynomials discussed previously in Section 3 are applied in this example, with $m = 14$, $j = 1, 2$, $k = 6$ and $r = 5$. In Table 1, the error of the computed solution obtained by the MB-polynomials was listed and compared with that of the B-polynomials. To evaluate the approximation values of the solution at a given x, the step-length $h = 0.2$ was taken.

Table 1: Absolute error on [0, 5] with m = 14 for Example 1.

x	y_i	B-polynomials	MB-polynomials
1.0	y_1	1.2155e−03	5.0000e−10
	y_2	1.1060e−06	5.5705e−12
2.0	y_1	1.9303e+07	1.1472e−15
	y_2	1.7564e+06	1.7157e−08
3.0	y_1	3.3651e+10	2.3656e−18
	y_2	3.0620e+08	3.3003e−08
4.0	y_1	4.1472e+12	5.1953e−20
	y_2	3.7735e+10	6.0985e−07
5.0	y_1	1.4710e+14	4.5545e−23
	y_2	9.3587e+15	2.1384e−07

Table 2: Absolute error on [0, 20] with m = 14 for Example 2.

x	y_i	BDF [7]	EBDF [9]	A-EBDF [9]	MB-polynomials
5.0	y_1	0.19e−03	0.16e−07	0.54e−08	0.10e−18
	y_2	0.15e−03	0.32e−07	0.13e−08	0.13e−19
10.0	y_1	0.90e−01	0.99e−10	0.46e−10	0.13e−20
	y_2	0.63e−01	0.18e−09	0.37e−10	0.44e−20
20.0	y_1	21672.865	0.11e−11	0.73e−14	0.62e−23
	y_2	1096.9431	0.83e−14	0.67e−13	0.59e−23

Example 2.

Consider the stiff system of ODE [9],

$$\begin{cases} y_1' = -y_1 - 15y_2 + 15\mathrm{e}^{-x}, \\ y_2' = 15y_1 - y_2 - 15\mathrm{e}^{-x}, \end{cases}$$

with the initial conditions $y_1(0) = 1, y_2(0) = 1$. The exact solution is

$$y_1 = y_2 = \mathrm{e}^{-x}.$$

B-polynomials are used and the MB-polynomials method discussed previously in Section 3 is applied in this example, with $m = 14$, $j = 1, 2$, $k = 6$ and $r = 20$. In Table 2 the error of the computed solution obtained by the MB-polynomials is listed and compared with that of the B-polynomials, BDF method [7] and EBDF method [9]. To evaluate the approximation values of the solution at a given x, the step-length was taken to be $h = 0.2$.

CONCLUSİONS

In this paper, the MB-polynomials method is considered a simple modification of the standard B-polynomials. MB-polynomials method was applied to solve stiff system of ODEs. Comparison between MB-polynomials and other methods such as B-polynomials, BDF, EBDF, and A-EBDF indicates that MB-polynomials can solve stiff problems more accurately with less iterations. MB-polynomials method gave more accuracy for large x than the standard B-polynomials.

REFERENCES

[1] M.H.T. Alshbool and I. Hashim. Multistage bernstein polynomials for the solutions of the fractional order stiff systems. Journal of King Saud University, 28:280–285, 2016.

[2] M.H.T. Alshbool, A.S. Bataineh, I. Hashim, and O. Isik. Approximate solutions of singular differential equations with estimation error by using Bernstein polynomials. International Journal of Pure and Applied Mathematics, 100(1):109–125, 2015.

[3] M.H.T. Alshbool, A.S. Bataineh, I. Hashim, and O. Isik. Solution of fractional-order differential equations based on the operational matrices of new fractional bernstein functions. Journal of King Saud University, 19:1–18, 2017.

[4] M.I. Bhatti and P. Bracken. Solutions of differential equations in a Bernstein polynomial basis. Computational and Applied Mathematics, 205:272–280, 2007.

[5] J. Biazar, M. Ali, and F. Salehi. Rational Homotopy perturbation method for solving stiff systems of ordinary differential equations. Applied Mathematical Modelling, 2014.

[6] J.R. Cash. On the integration of stiff system of ODEs using extended backward differentiation formula. Numerische Mathematik, 34:235–246, 1980.

[7] J.R. Cash. The integration of stiff initial value problems in ODEs using modified extended backward differentiation formula. Computers and Mathematics with Applications, 9:645–657, 1983.

[8] J.H. He. Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons and Fractals, 26:695–700, 2005.

[9] G. Hojjati, M.Y. Rahimi, and S.M. Hosseini. A-EBDF: an od for numerical solution of stiff system of ODEs. Mathematics and Computer in Simulation, 66:33–41, 2004.

[10] K. Pandey, A. Bhardwaj, and I. Syam. An efficient method for solving fractional differential equations using Bernstein polynomials of order eight. Journal of Fractional Calculus and Applications, 5(1):129–145, 2014.

[11] S.A. Yousefi and M. Behroozifar. Operational matrices of Bernstein polynomials and their applications. International Journal of System Science, 41:709–716, 2010.

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