MINIMIZATION OF THE FUZZY VARIANCE FOR THE PORTFOLIO SELECTION: THEORY AND APPLICATION ON THE TURKISH STOCKS

[1]CELAL BARKAN GURAN, [2]OKTAY TAS, [3]AYSUN GURAN

Istanbul Technical University , Istanbul Technical University, Doguş University
E-mail: [1]barkan_guran@hotmail.com, [2]oktay.tas@itu.edu.tr, [3]adogrusoz@dogus.edu.tr

Abstract- In today's financial world portfolio management has become a big sector where every investor tries to increase the return and decrease the risk at the same. The classical Mean Variance (MV) portfolio optimization method which is proposed by H. Markowitz in 1952 has some limitations which do not satisfy today's financial needs when working with real data. In this study, a portfolio optimization model is introduced which is based on fuzzy variance minimization at a given return level. This method brings a fuzzy approach to the MV defined by Markowitz so that the future returns of the stocks are predicted with the help of the triangular fuzzy numbers. As an empirical example, this method is applied to the 12 well known stocks of the Turkish Market. For further research, this portfolio optimization method which adds the fuzzy logic perspective to MV will be applied to the other markets with various performance tests.

Index Terms- Fuzzy Variance Minimization, Portfolio Optimization, Triangular Fuzzy Numbers, Turkish Bist-30 Index.

I. INTRODUCTION

Nowadays investors are constantly faced with the dilemma of guessing the direction of future market moves in order to meet the return target for assets. Because of the uncertainty inherent in financial markets, financial experts must be very cautious in expressing their market views. The information content in such circumstances can be best described as being "fuzzy", in terms of both the direction and the size of market moves. Nevertheless, this is one of the best ways to structure portfolios so that the target return, which is assumed to be higher than the risk-free rate, is met. In general, achieving returns higher than the risk-free rate requires taking either market or credit risk.

H. Markowitz proposed the classical Mean Variance (MV) portfolio optimization method in 1952 and he won the Nobel Prize with this work. But this optimization method has serious constraints which brings problems especially when working with real data of today's financial world. The requirement that all returns must be normally distributed is just one of the examples of these unsatisfied restrictions. To overcome this problem a new perspective supported by fuzzy logic is added to the classical MV portfolio optimization method which minimizes the "fuzzy variance" instead of the "classical variance". In addition to its theoretical background, this method is also applied to the 12 well known stocks of the Turkish Stock Exchange (BIST).

The remaining parts of the paper are organized as follows: Section II explains literature review; Section III outlines theory of the Model; Section IV points out application of the model and finally Section V gives concluding remarks.

II. LITERATURE REVIEW

Under certain assumptions and for specific quantitative definitions of risk and return, Modern Portfolio Theory explains how to find the best possible diversification strategy. Markowitz (1952) proposed Mean Variance (MV) method and later he wrote a book. (Markowitz, 1959) where he minimized the portfolio variance. MV is very common because of its simple algorithm which allows finding the optimal weights. But on the other side, in real life there are some very important limitations of MV.

Despite its theoretical importance, critics of MV Optimization question whether it is an ideal investing strategy, because its model of financial markets does not match the real world in many ways. Brodie et al (2009) has shown recently that instabilities of this type disappear when a regularizing constraint or penalty term is incorporated in the optimization procedure.

MV Optimization requires that the returns are normally distributed, but in the real data it is very rare to find normal distributed returns. Samuelson (1970) and Constandinides and Malliaris (1995) discuss this topic in detail and they work on the choice of MV optimal portfolios.

MV Optimization deals only with two parameters - mean and variance - but there are two other significant parameters such as skewness and kurtosis. There are many researches showing that risk averse investors prefer positive skewness and avoid kurtosis. For further details of MV Optimization's shortcomings the readers can examine the studies of Kraus and Litzenberger (1976), Athayde and Flores (1997), Fang and Lai (1997), Dittmar (2002), Post, Levy and Vliet (2008), Wong (2007).

As a fuzzy approach Carlsson and Fullër (2001) defined possibilistic mean value, variance and covariance of fuzzy numbers by developing earlier works of Dubois and Prade (1987). According to these works, A fuzzy number A is a fuzzy set of the real line R with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by F. Let A be a fuzzy number with γ level set $[A]_\gamma = [a_1(\gamma), a_2(\gamma)](\gamma > 0)$. So the possibilistic mean value of fuzzy numbers is as below:

$$M(A) := \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \frac{\int_0^1 \gamma \cdot \frac{a_1(\gamma)+a_2(\gamma)}{2} d\gamma}{\int_0^1 \gamma\, d\gamma}, \quad (1)$$

It follows that M(A) is nothing else but the level-weighted average of the arithmetic means of all γ level sets, that is, the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just γ. In a similar manner possibilistic variance of fuzzy numbers can be defined as below:

$$\begin{aligned}\mathrm{Var}(A) &= \int_0^1 \mathrm{Pos}[A \le a_1(\gamma)] \left(\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_1(\gamma) \right]^2 \right) d\gamma \\ &\quad + \int_0^1 \mathrm{Pos}[A \ge a_2(\gamma)] \left(\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_2(\gamma) \right]^2 \right) d\gamma \\ &= \int_0^1 \gamma \left(\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_1(\gamma) \right]^2 \right. \\ &\quad \left. + \left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_2(\gamma) \right]^2 \right) d\gamma \\ &= \frac{1}{2} \int_0^1 \gamma (a_2(\gamma) - a_1(\gamma))^2 d\gamma. \end{aligned} \quad (2)$$

Let apply these general definitions where a1(γ) and a2(γ) are linear functions in other words let them apply to a triangular fuzzy number (a,α,β) with center a, left with center a, leftwidth α > 0 and right-width β > 0 as depicted in Figure 1:

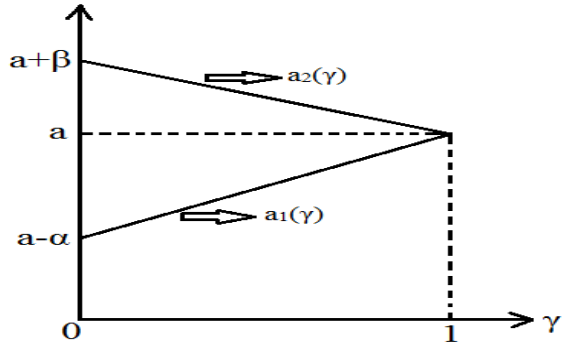


Figure 1. Triangular fuzzy number (a,α,β) with center a and presentation of a1(γ) and a2(γ).

Because of their linear nature by using simple mathematic rules ai(γ), i=1,2 become

$a_1(\gamma)=a-(1-\gamma)\alpha$, $a_2(\gamma)=a+(1-\gamma)\beta$; $\gamma \in [0,1]$ (3)

When (3) is put on (1), possibilistic mean of the triangular fuzzy number A=(a,α,β) is calculated after a few integral calculations as below

$$M(A) = a + \frac{\beta - \alpha}{6} \quad (4)$$

After putting (Eq. 3) into (Eq. 2) possibilistic variance A=(a,α,β) is calculated in a similar manner after a few more integral calculations as below:

$$\mathrm{Var}(A) = [\![(\alpha+\beta)]\!]\,^2/24 \quad (5)$$

By considering two fuzzy numbers A and B such that $[A]_\gamma=[a_1(\gamma), a_2(\gamma)]$ and $[B]_\gamma=[b_1(\gamma), b_2(\gamma)]$ ($\gamma > 0$) the possibilistic covariance of A and B can be defined as below:

$$\mathrm{Cov}(A,B) = \frac{1}{2}\int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))(b_2(\gamma) - b_1(\gamma))d\gamma \quad (6)$$

When A=(a,α,β) and B=(b, θ, λ) are triangular fuzzy numbers then the related functions become as below:

$a_1(\gamma)=a-(1-\gamma)\alpha$, $a_2(\gamma)=a+(1-\gamma)\beta$

$b_1(\gamma)=b-(1-\gamma)\theta$, $b_2(\gamma)=b+(1-\gamma)\lambda$ (7)

If (Eq.7) is put on (Eq.6) then Cov(A,B) can calculated as:

$$Cov(A,B) = \frac{(\alpha+\beta)(\theta+\lambda)}{24} \quad (8)$$

It can be noticed when A=B then Cov(A,A) is equals to (α+β)2/24 which is Var(A) as expected according to statistics rules. Apart from these definitions, let λ, μ ∈ R and let A and B be fuzzy numbers. Then the expression can be written as below:

$$\mathrm{Var}(\lambda A + \mu B) = \lambda^2\,\mathrm{Var}(A) + \mu^2\,\mathrm{Var}(B) + 2|\lambda\mu|\,\mathrm{Cov}(A,B) \quad (9)$$

where the addition and multiplication by a scalar of fuzzy numbers is defined by the sup-min extension principle. Carlsson and Fullër (2001) proved this theorem which forms the fundamental of the portfolio variance in fuzzy environment. If this theorem is generalized to n fuzzy numbers $A_1,...,A_n$ with constants $c_1,...,c_n \in R$, the generalized result can be obtained as:

$$\begin{aligned}&\mathrm{Var}(c_1A_1+\ldots+c_nA_n)=c_1^2\mathrm{Var}(A_1)+\ldots+c_n^2\mathrm{Var}(A_n)+ \\ &2|c_1c_2|\mathrm{Cov}(A_1,A_2)+\ldots+2|c_{n-1}c_n|\mathrm{Cov}(A_{n-1},A_n)\end{aligned} \quad (10)$$

III. THEORY OF THE MODEL

The optimized portfolio is found by minimizing the fuzzy variance for a given target return level. In terms of fuzzy logic, triangular membership functions are used to forecast future returns both because of its suitable nature to the portfolio selection problem and because of its linear structure which facilitates the optimization model. To predict the future return of a stock, it is assumed that the membership degree of the fuzzy average is always 1. But the membership degrees will change depending on the scenario whilst deviating vastly from the fuzzy average. The triangular membership function representing the future returns r_i of the i^{th} stock (i=1,...,n) :

$$\mu_i(u) = \begin{cases} \frac{1}{\alpha_i}u + \frac{\alpha_i - r_i}{\alpha_i}, & r_i - \alpha_i \le u \le r_i \\ -\frac{1}{\beta_i}u + \frac{\beta_i + r_i}{\beta_i}, & r_i \le u \le r_i + \beta_i \\ 0, & otherwise \end{cases} \quad (11)$$

In fuzzy terms this triangular membership function can be expressed by $(r_i-\alpha_i, r_i, r_i+\beta_i)$ where β_i and α_i represent maximum possible differences of future returns respectively in up and down directions and r_i is

the expected center future return with the highest membership value of 1. After the calculation of r_i, α_i and β_i values of each stock, the fuzzy mean value of the whole portfolio return of n assets can be defined as:

$$M(r) = \sum_{i=1}^{n} M(r_i)x_i \quad (12)$$

In this definition x_i (i=1,...,n) represents the weights of the stocks. These proportions satisfy the condition as below because in the proposed models short selling of any stock is not allowed.

$$\sum_{i=1}^{n} x_i = 1, \quad 0 \le x_i \le 1, i = 1, \dots, n \quad (13)$$

Since r_i is the midpoint of the triangular fuzzy number for stock i, by using (Eq. 4) the $M(r_i)$ can be expressed as:

$$M(r_i) = r_i + \frac{\beta_i - \alpha_i}{6}, i = 1, \dots, n \quad (14)$$

If (Eq. 14) is put into (Eq. 12) the equation below can be obtained for the fuzzy mean value of the whole portfolio return.

$$M(r) = \sum_{i=1}^{n} (r_i + \frac{\beta_i - \alpha_i}{6})\, x_i \quad (15)$$

According to portfolio theory, to solve the right portfolio selection problem, in other words to find the optimal weights, the fuzzy variance of the whole portfolio must be defined in addition to fuzzy portfolio return. For this purpose, the identity in (Eq.10) can be used to determine the portfolio variance as below:

$$Var(r) = Var(\sum_{i=1}^{n} r_i\, x_i) = x_1^2\, Var(r_1) + \dots + x_n^2 Var(r_{10}) + 2|x_1x_2|Cov(r_1,r_2) + \dots + 2|x_{n-1}x_n|Cov(r_{n-1},r_n) \quad (16)$$

Because of the triangular feature of r1,...,rn , if (Eq.5) and (Eq.8) are applied to (Eq.16) the formula as below can be concluded to determine the portfolio variance.

$$Var(r) = \sum_{i=1}^{n} \frac{(\alpha_i+\beta_i)^2}{24} x_i^2 + 2\sum_{i\neq j=1}^{n} \frac{(\alpha_i+\beta_i)^2(\alpha_j+\beta_j)^2}{24} x_i x_j = [\sum_{i-1}^{n} (\frac{\alpha_i+\beta_i}{2\sqrt{6}}) x_i]^2 \quad (17)$$

The fuzzy mean-variance model can formulated as below :

$$Min \left\{ [\sum_{i=1}^{n} (\frac{\alpha_i + \beta_i}{2\sqrt{6}}) x_i]^2 \right\}$$

$$\text{Subject to } \sum_{i=1}^{n} (r_i + \frac{\beta_i - \alpha_i}{6})\, x_i \ge \mu,$$

$$\sum_{i=1}^{n} x_i = 1,$$

$$0 \le x_i \le 1, i = 1, \dots, n \quad (18)$$

In this quadratic nonlinear optimization model, μ represents the desired minimum portfolio return level and this size of μ is left to the preference of the portfolio maker. It is obvious that the greater the risk, the greater the appetite of the investor and the bigger the size of the desired minimum portfolio return.

To facilitate the solution of this optimization model, standard deviation can be used instead of variance. When the square root of the objective function is taken, all nonlinear terms drop from the formula and the defuzzified standard deviation of the whole portfolio is as below:

$$\sqrt{Var(r)} = Std\ Dev(r) = \frac{\sqrt{6}}{12} \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i \quad (19)$$

When defuzzified standard deviation is minimized instead of fuzzy variance, the model in (Eq. 18) is transformed to a linear optimization model as below:

$$Min \left\{ \frac{\sqrt{6}}{12} \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i \right\}$$

$$\text{Subject to } \sum_{i=1}^{n} (r_i + \frac{\beta_i - \alpha_i}{6})\, x_i \ge \mu,$$

$$\sum_{i=1}^{n} x_i = 1,$$

$$0 \le x_i \le 1, i = 1, \dots, n \quad (20)$$

IV. APPLICATION OF THE MODEL

To apply this model, the most popular 12 stocks of the Turkish Stock Exchange is taken. Table 1 shows these stocks. All data consisting of the closing values of these stocks from 03.12.2010 to 05.07.2013 are taken from the official web site of Istanbul Stock Exchange, borsaistanbul.com, on a weekly basis total of 135 observation periods. The data before 03.12.2010 would be incomplete since some of these 12 companies have no returns before 03.12.2010.

Table 1. The list of stocks of the application

	Name of the Company	Stock Code	Market Value (USD)
1	ARÇELIK A.Ş.	ARCLK	3.702.791.086
2	ENKA İNŞAAT VE SANAYİ A.Ş	ENKAI	7.380.073.801
3	EREGLI DEMIR VE ÇELIK FABRIKALARı T.A.Ş.	EREGL	5.269.372.694
4	İPEK DOĞAL ENERJI KAYNAKLARı ARAŞTıRMA VE ÜRETIM A.Ş.	IPEKE	266.495.890
5	KOÇ HOLDING A.Ş.	KCHOL	11.743.734.512
6	KARDEMIR KARABÜK DEMIR ÇELIK SANAYI VE TICARET A.Ş.	KRDMD	437.617.536
7	PETKIM PETROKIMYA HOLDING A.Ş.	PETKM	1.523.985.240
8	HACı ÖMER SABANCı HOLDING A.Ş.	SAHOL	7.393.640.813
9	TAV HAVALIMANLARı HOLDING A.Ş.	TAVHL	2.949.146.679
10	TURKCELL İLETIŞIM HIZMETLERI A.Ş.	TCELL	10.228.782.288
11	TÜRK HAVA YOLLARı A.O.	THYAO	4.715.424.354
12	TÜRK TELEKOMÜNIKASYON A.Ş.	TTKOM	8.898.523.985

The fuzzy variance minimization is applied to the 12 stocks. To apply the fuzzy variance minimization, first of all, the membership function displaying the future return should be determined for each of these 12 stocks. To determine these functions, it is easy to compute the expected center future return (r_i) since it can be accepted as the average return of the stock (i=1,...,12). But there is not just a certain rule to estimate the up direction (β_i) and the down direction (α_i) representing the maximum possible differences of future returns. Actually these deviations reflect the expert knowledge. That means this fuzzy model enables that up and down directions can be determined according to the next coming economic conditions. But in this application part of the thesis no subjective opinion is added to the model. The past observations of the stocks in the last 135 weeks are observed and under the lights of this information β_i and α_i are determined due to relatively the best and the worst returns in the past. The logic of βi and αi calculation can be explained by the "Arçelik" Stock which belongs to BIST-30 Portfolio. Figure 2 shows the related graph of this example.

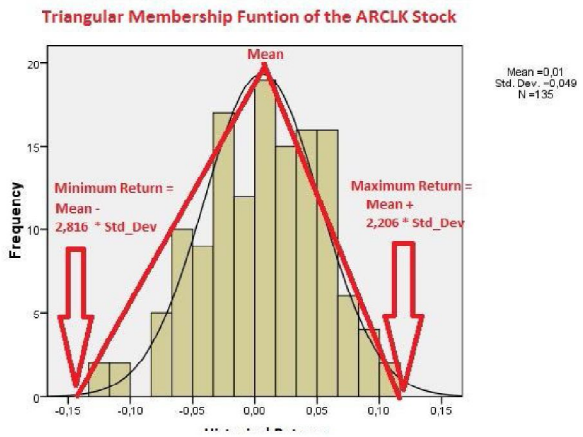


Figure 2. The calculation logic of up (β_i) and down (α_i) directions explained by the "Arçelik" Stock.

As it is seen from Figure 2 the best and the worst returns of the stock constitute the end points of the triangular base side. Table 2 shows the necessary calculations of the up and down deviations which belong to the 12 stocks in order to apply the optimization model. The linear optimization model can be obtained as shown in Table 3 with the help of the β_i and α_i values. A simulation with various return levels is done in the solution step of this optimization method since the minimum goal return in the first constraint is a changing variable. In the final step, the solver tool of the Excel is used to solve this linear optimization method by different risk free return levels. The related fuzzy standard deviation, risk free return and the weights of the optimized portfolios are displayed in Table 4. As it can be directly realized, there is no feasible solution of the model with the goal returns larger than 0.045.

Table 2. The necessary calculations of the SSD efficient 12 stocks to minimize the fuzzy variance.

12 SSD Efficient Stocks	ARCLK	ENKAI	EREGL	IPEKE	KCHOL	KRDMD	PETKM	SAHOL	TAVHL	TCELL	THYAO	TTKOM
Average	0,006	0,002	-0,001	0,006	0,004	0,006	0,002	0,003	0,004	0,001	0,005	0,003
Risk Free Average	0,005	0,000	-0,002	0,005	0,002	0,005	0,001	0,002	0,003	0,000	0,004	0,002
Std Dev	0,049	0,046	0,042	0,078	0,045	0,045	0,039	0,046	0,039	0,036	0,049	0,038
Min Deviation= Average+ (x)* Std Dev	-2,816	-2,417	-2,582	-2,979	-2,524	-3,821	-3,400	-2,888	-3,293	-4,143	-2,757	-3,170
Max Deviation= Average+ (x)* Std Dev	2,206	3,233	2,413	6,334	3,620	2,386	3,586	2,939	2,478	2,833	3,139	3,070
β (deviation in up direction)	0,108	0,149	0,101	0,494	0,164	0,108	0,140	0,135	0,096	0,101	0,154	0,118
α (deviation in down direction)	0,138	0,112	0,108	0,232	0,114	0,172	0,133	0,133	0,128	0,148	0,135	0,122
[sqrt(6)/12]*(α+β)	0,050	0,053	0,043	0,148	0,057	0,057	0,056	0,055	0,046	0,051	0,059	0,049
Average+(β-α)/6	0,001	0,008	-0,002	0,050	0,012	-0,005	0,003	0,004	-0,001	-0,006	0,008	0,002
Risk Free Average + (β-α)/6	0,000	0,007	-0,003	0,049	0,011	-0,006	0,002	0,002	-0,002	-0,008	0,007	0,001

Table 3. The linear optimization model.

Min [0,050 x1 + 0,053 x2 + 0,043 x3 + 0,148 x4 + 0,057 x5 + 0,057 x6 + 0,056 x7 + 0,055 x8 + 0,046 x9 + 0,051 x10 + 0,059 x11 + 0,049 x12]
Subject to
-0,0005 x1 + 0,0068 x2 - 0,0031 x3 + 0,0489 x4 + 0,0107 x5 - 0,0062 x6 + 0,0022 x7 + 0,0025 x8 - 0,0024 x9 - 0,0075 x10 + 0,0070 x11 + 0,0010 x12 >= "Goal Return"
x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 = 1
x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12 >= 0

Table 4. Fuzzy Varince minimization of the 12 stocks.

		Weights											
		ARCLK	ENKAI	EREGL	IPEKE	KCHOL	KRDMD	PETKM	SAHOL	TAVHL	TCELL	THYAO	TTKOM
Fuzzy Logic Std Dev	Risk Free Return	1	2	3	4	5	6	7	8	9	10	11	12
0,0487	0,0028	0	0	0,5711	0	0,4289	0	0	0	0	0	0	0
0,0491	0,0032	0	0	0,5420	0	0,4580	0	0	0	0	0	0	0
0,0495	0,0036	0	0	0,5130	0	0,4870	0	0	0	0	0	0	0
0,0499	0,004	0	0	0,4840	0	0,5160	0	0	0	0	0	0	0
0,0504	0,0045	0	0	0,4477	0	0,5523	0	0	0	0	0	0	0
0,0508	0,0049	0	0	0,4187	0	0,5813	0	0	0	0	0	0	0
0,0512	0,0053	0	0	0,3897	0	0,6103	0	0	0	0	0	0	0
0,0518	0,0058	0	0	0,3534	0	0,6466	0	0	0	0	0	0	0
0,0524	0,0064	0	0	0,3099	0	0,6901	0	0	0	0	0	0	0
0,0560	0,01	0	0	0,0487	0	0,9513	0	0	0	0	0	0	0
0,0671	0,015	0	0	0	0,1132	0,8868	0	0	0	0	0	0	0
0,0790	0,02	0	0	0	0,2440	0,7560	0	0	0	0	0	0	0
0,0910	0,025	0	0	0	0,3748	0,6252	0	0	0	0	0	0	0
0,1029	0,03	0	0	0	0,5056	0,4944	0	0	0	0	0	0	0
0,1149	0,035	0	0	0	0,6364	0,3636	0	0	0	0	0	0	0
0,1269	0,04	0	0	0	0,7672	0,2328	0	0	0	0	0	0	0
0,1388	0,045	0	0	0	0,8980	0,1020	0	0	0	0	0	0	0
0,1481	0,05												
0,1481	0,06												
0,1481	0,07												
0,1481	0,08												

No Feasible Solution

CONCLUSION AND FUTURE WORK

In this paper a portfolio optimization method which minimizes the fuzzy variance of the portfolio is introduced and this method is applied to the 12 well known stocks of Turkish Stock Market. With its fuzzy background this method has some serious advantages compared to the classical MV optimization which is introduced by Markowitz 63 years ago. Firstly, the portfolio managers can add their subjective opinions to the model with the help of triangular fuzzy numbers representing the future returns of the stocks. Secondly, the model carries computation simplicity since only linear terms remains in its objective function because of the linear character of the used membership functions. Thirdly, this method does not require the limitations of classical MV optimization which are listed in detail at the beginning of this paper.
The aim in the short run is to measure the performance of this method by comparing it with the results of other methods. The aim in the long run is to apply this new method to other markets of the world apart from Turkish Stock Exchange and generalize its results for the whole finance world.

REFERENCES

[1] Athayde, G. & Flores, R. (1997). Introducing Higher Moments in CAPM. Some basic Ideas. Working Paper.

[2] Brodie, J., De Mol, C., Daubechies, I., Giannone, D. & Loris, I. (2009). Sparse and stable Markowitz portfolios, Proceedings of the National Academy of Sciences, 106 (30).

[3] Carlsson C. & Fullër R. (2001). On possibilistic mean value and variance of fuzzy numbers. Fuzzy Sets and Systems, 122, 315–326.

[4] Constandinides, G. M. & Malliaris, A. G. (1995). Portfolio theory. In. Jarrow, R., et al. (Eds.), Handbooks in Operations Research and Management Science, Elsevier Science B.V.,Amsterdam, 9, 1–30.

[5] Dittmar, R. (2002). Nonlinear asset kernels kurtosis preference and evidence from cross section of equity returns. Journal of Finance, 57, 369-403.

[6] Dubois D. & Prade H. (1987). The mean value of a fuzzy number, Fuzzy Sets and Systems, 24, 279-300.

[7] Fang, H. & Lai, T. (1997). Co-kurtosis and capital asset pricing. Financial Review, 32, 293–307.

[8] Kraus, A. & Litzenberger, R. (1976). Skewness preference and the valuation of risky assets. Journal of Finance, 31, 1085-1099.

[9] Markowitz, H. M. (1952). Portfolio Selection. The Journal of Finance 7 (1), 77–91.

[10] Markowitz, H. M. (1959). Portfolio Selection. Efficient Diversification of Investments. New York. John Wiley & Sons.

[11] Post, T., Vlient, P., & Levy, H. (2008). Risk aversion and skewness preference. A comment. Journal of Banking and Finance, 32, 1178-1187.

[12] Samuelson, P. A. (1970). The fundamental approximation theorem of portfolio analysis in terms of means, variances, and higher moments. Review of Economical Studies, 36, 537–542.

[13] Wong, W. K. (2007). Stochastic dominance and mean–variance measures of profit and loss for business planning and investment, European Journal of Operational Research, 182, 829–843.

★★★