# MODEL PREDICTIVE CONTROL USING CONTRACTION MAPPING NUMERICAL SOLUTION ALGORITHM FOR AUTONOMOUS OVERTAKING DRIVING

**[1]TASSEI UEMATSU, [2]TOMOAKI HASHIMOTO**

Osaka Institute of Technology, Osaka Institute of Technology
E-mail: [1]uematsu.tassei@sys-lab.org, [2]tomoaki.hashimoto@oit.ac.jp

**Abstract** - This paper examines the design problem of control system for autonomous overtaking driving. Model predictive control (MPC) method is a kind of optimal feedback control method in which the control performance over a finite future is optimized. The contraction mapping method is a fast numerical solution method used for solving nonlinear model predictive control problems. The objective of this paper is to propose a nonlinear model predictive control method using a fast numerical solution algorithm called contraction mapping method for autonomous overtaking driving systems. The effectiveness of the proposed method is verified by numerical simulation.

**Keywords** – Autonomous Vehicle, Model Predictive Control, Nonlinear Dynamics, Optimal Control

## I. INTRODUCTION

In recent decades, a variety of control problems of autonomous vehicle such as collision avoidance [1], rollover prevention [2], wheel slip control [3], driver assistance control [4] have been investigated. Particularly, model predictive control (MPC) method is widely applied to solving several vehicle control problems. In [5], a linear MPC has been devised for control of steering and braking in autonomous vehicle navigation. In [6], MPC method has been devised to solve the problem with brake torque constraints of electronic mechanical brake mechanism for a vehicle yaw stability. Furthermore, MPC method based on an integrated control algorithm for vehicle in active steering and dynamics yaw control has been devised in [7].

MPC is a well-established control method in which the control input is obtained by solving an open-loop optimal control problem with finite evaluation interval. This optimization procedure is repeated at each sampling instant. Thus, MPC is a type of optimal feedback control in which the control performance over a finite future is optimized and its performance index has a moving initial time and a moving terminal time. MPC is considered as one of the most successful control methodologies because it enables control performance to be optimized while taking constraints on state and control variables into consideration [8]-[12].

The model predictive control problem of vehicle nonlinear dynamics is considered as nonlinear model predictive control (NMPC) problem that needs to be solved by numerical solution algorithm. Thus, the implementation of NMPC, in which the control problem is solved on-line, poses significant challenges in terms of computational load. In [13], the NMPC problem has been devised for stabilization of vehicle nonlinear dynamics to avoid the second accident after the first collision accident. Then, a fast numerical solution method has been developed in [13] based on C/GMRES algorithm [14] to solve the NMPC problem for stabilization of unstable vehicle dynamics caused by collision accident.

Recently, numerical solution algorithm called contraction mapping method has been proposed in [15]. It has been shown that this algorithm is faster than the C/GMRES algorithm [14] from the computational point of view. This study examines the design problem of control system for autonomous overtaking driving. The NMPC problem for autonomous overtaking driving needs to be solved within a short sampling period.

The objective of this paper is to propose a nonlinear model predictive control method using a fast numerical solution algorithm called contraction mapping method for designing an autonomous overtaking driving control system. The effectiveness of the proposed method is verified by numerical simulation.

## II. VEHICLE SYSTEM MODEL

In this section, a system model of vehicle nonlinear dynamics is introduced under the following assumptions. First, it is assumed that the difference between the vertical loads of the left and right wheels is negligible. Second, it is assumed that rolling and pitching motions are negligible. Finally, it is assumed that the rear tires are not steered. Under those assumptions, the cornering forces of the left and right wheels are equal each other. Thus, four-wheeled vehicle dynamics can be considered as two-wheeled vehicle dynamics. In this study, a two-wheeled vehicle model shown in Fig. I, is equivalent to a four-wheeled vehicle model. The system parameters used in this model are listed in Table I.

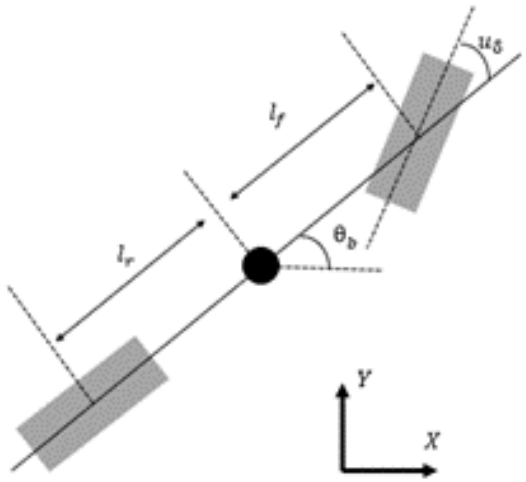


**Fig. I. Two-wheeled vehicle model equivalent to four-wheeled vehicle model**

| Notation | Definition |
|---|---|
| $m_b$ | mass of vehicle |
| $a_{c1}$ | acceleration of the front vehicle |
| $\tau_b$ | delay in braking force |
| $\rho$ | air density |
| $C_b$ | air resistance coefficient |
| $I_Z$ | yaw moment of inertia |
| $l_f$ | distance from C.G. to front axle |
| $l_r$ | distance from C.G. to rear axle |
| $C_f$ | cornering stiffness – Front tire |
| $C_r$ | cornering stiffness – Rear tire |
| $A_b$ | forward projection area |
| $\theta$ | gradient |
| $g$ | acceleration of gravity |
| $p$ | curvature |
| $\mu$ | Rolling resistance coefficient |
| $v_{c1}$ | velocity of vehicle in front |
| $d$ | distance between two vehicles |
| $y$ | vehicle lateral displacement |
| $\dot{y}$ | lateral velocity in vehicle coordinate |
| $\theta_b$ | vehicle angle |
| $\dot{\theta}_b$ | vehicle yaw rate |
| $a_b$ | rate of acceleration with delay |
| $v_b$ | velocity |
| $x_b$ | vehicle longitudinal displacement |
| $u_a$ | rate of acceleration |
| $u_\delta$ | steering angle of front wheel |

**Table I: System parameters**

For notational simplicity, we introduce the state and input vectors as follows:

$$x(t) = \left[v_{c1}, d, y, \dot{y}, \theta_b, \dot{\theta}_b, a_b, v_b, x_b\right]^{\mathrm{T}}$$
$$u(t) = [u_a\ , u_\delta]^{\mathrm{T}}$$

Using these notations, the system model of vehicle dynamics can be described by the following state equation:

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$

$$\begin{aligned} f\big(x(t),u(t)\big) \coloneqq \Big[ & a_{c1}x_1 - x_8\cos(x_5)\,x_4 - \frac{a_{11}}{x_8}x_4 \\ & + a_{11}x_5 + \frac{a_{12}}{x_8}x_6 + b_1u_2 \\ & + (a_{12} - {x_8}^2)px_6 - \frac{a_{21}}{x_8}x_4 \\ & + a_{21}x_5 + \frac{a_{22}}{x_8}x_6 + b_2u_2 \\ & + (a_{22} - x_7)p\frac{1}{\tau_b}(-x_7 + u_1) \\ & - \frac{1}{m_b}\Big(\frac{1}{2}\rho A_b C_b {x_8}^2 + m_b g sin\theta \\ & + \mu m_b g cos\theta\Big) \\ & + x_7x_8\cos(x_5)\Big] \#(1) \end{aligned}$$

$$a_{11} = \frac{2\big(C_f + C_r\big)}{m_b}, a_{12} = -\frac{2\big(l_fC_f + l_rC_r\big)}{m_b}$$
$$a_{21} = \frac{2\big(l_fC_f + l_rC_r\big)}{I_z}, a_{22} = \frac{2\big({l_f}^2C_f + {l_r}^2C_r\big)}{I_z}$$
$$b_1 = \frac{2C_f}{m_b}, b_2 = \frac{2l_fC_f}{I_z}$$

Hereafter, we consider equation (1) as the system model of vehicle motion with nonlinear dynamics

## III. NONLINEAR MODEL PREDICTIVE CONTROL

In this section, the nonlinear model predictive control problem of system model (1) is considered. First, the optimal control problem of nonlinear vehicle dynamics is considered. The control input at each time $t$ is determined so as to minimize the following performance index:

$$J = \varphi\big(x(t+T)\big) + \int_t^{t+T} L\big(x(\tau), u(\tau)\big)d\tau \#(2)$$

where $T$ is the evaluation interval of the performance index, $\varphi$ is the terminal cost function and $L$ is the stage cost function. The optimization problem of (2) subject to equality constraint (1) can be reduced to minimizing the following performance index $\underline{J}$ introduced by using the costate $\lambda$ associated with the equality constraint.

$$\begin{aligned} \underline{J} = \varphi\big(x(t+T)\big) + \int_t^{t+T} & \Big(L\big(x(\tau),u(\tau)\big) \\ & + \lambda^T(\tau)\big(f(x,u) - \dot{x}\big)\Big)d\tau \end{aligned}$$

Let the Hamiltonian $H$ be defined by

$$\begin{aligned} & H\big(x(\tau),u(\tau),\lambda(\tau)\big) \\ & = L\big(x(\tau),u(\tau)\big) \\ & + \lambda^T(\tau)f\big(x(\tau),u(\tau)\big) \end{aligned}$$

Then, the performance index $\underline{J}$ is rewritten by

$$\underline{J} = \varphi\big(x(t+T)\big) + \int_t^{t+T} \Big(H\big(x(\tau),u(\tau),\lambda(\tau)\big) - \lambda^T(\tau)\dot{x}(\tau)\Big)\,d\tau$$

Next, we consider the variation in $\underline{J}$ as follows:

$$\delta\underline{J} = \frac{\partial\varphi}{\partial x}\delta x(t+T) + \int_t^{t+T} \left(\left(\frac{\partial H}{\partial x}\right)\delta x + \left(\frac{\partial H}{\partial u}\right)\delta u + \delta\lambda^T\left(\frac{\partial H}{\partial \lambda} - \dot{x}\right) - \lambda^T\delta\dot{x}\right)d\tau$$

It is worth noting that we apply the following integration by parts to the computation on $\delta\underline{J}$.

$$\int_t^{t+T} \big(\lambda^T(\tau)\,\delta\dot{x}(\tau)\big)d\tau = [\lambda^T(\tau)\,\delta x(\tau)]_t^{t+T} - \int_t^{t+T} \Big(\dot{\lambda}^T(\tau)\,\delta x(\tau)\Big)\,d\tau$$

In the above, note that we take $\delta x(t) = 0$ because $x(\tau)$ is fixed at $\tau = t$. Taking the above integration by parts into account, we obtain the variation in $\underline{J}$ as

$$\delta\underline{J} = \left(\frac{\partial\varphi}{\partial x} - \lambda^T(t+T)\right)\delta x(t+T) + \int_t^{t+T} \left(\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T(\tau)\right)\delta x + \left(\frac{\partial H}{\partial u}\right)\delta u + \delta\lambda^T\left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)\right)d\tau$$

On the basis of the variational principle, we obtain the necessary conditions for a stationary value of $\underline{J}$ over the horizon $(t \le \tau \le t+T)$ as follows.

$$\dot{x}(\tau) = f\big(x(\tau),u(\tau)\big)\#(3)$$
$$\dot{\lambda}(\tau) = -\left(\frac{\partial H}{\partial x}\right)^T\#(4)$$
$$\lambda(t+T) = \left(\frac{\partial\varphi}{\partial x}\right)^T\#(5)$$
$$\frac{\partial H}{\partial u} = 0\#(6)$$

Conditions (3)-(6) are called the stationary conditions or Euler-Lagrange equations that must be satisfied for the performance index (2) to be minimized. A well-known difficulty of the nonlinear optimal control is that it results in a nonlinear two-point boundary-value problem that cannot be solved analytically in general. Then, a fast algorithm, called the contraction mapping method, for numerically solving stationary conditions has been proposed in [15]. A brief description of the contraction mapping method applied to this problem is presented in the subsequent discussion.

To solve the stationary conditions in (3)-(6) using numerical algorithm, we must discretize equations (3)-(6) into discrete-time equations. Let the time $\tau \in [t, t+T]$over the prediction horizon be divided into $N$ steps and $k$ denote the discretized temporal variable. As a result of the discretization approximation, we obtain the discretized stationary conditions over the horizon $(k = 1 \cdots N)$ as follows:

$$x(k+1) = f\big(x(k),u(k)\big)\#(7)$$
$$\lambda(k) = g\big(x(k+1),u(k+1),\lambda(k+1)\big)\#(8)$$
$$\lambda(N) = \frac{\partial\varphi}{\partial x}\big(x(N)\big)\#(9)$$
$$\frac{\partial H}{\partial u}\big(x(k),u(k),\lambda(k)\big) = 0\#(10)$$

Note that $x(1)$is identical to the present known state $x(t)$. The time-evolutionary equations of $x$ and $\lambda$ are discretized into forward difference equation in (7) and backward difference equation in (8), respectively. Let the optimization $u(k), (k = 1 \cdots N)$be unified into a vector denoted by

$$U(t) := [u^T(1) \ \cdots \ u^T(N)]^{\mathrm{T}}$$

In the following, we provide the procedure for computing $x, \lambda, u$ to satisfy stationary conditions (7)-(10). For the present state $x(1)$ and a given initial solution $U(t)$, $x(k)$ for $(k = 1 \cdots N)$ is calculated recursively from $k = 1$ to $k = N$ by using (7). Next, the terminal costate $\lambda(N)$ is determined from the terminal state $x(N)$ by using (9). Consequently, $\lambda(k)$ for $(k = 1 \cdots N)$ is calculated recursively from $k = N$ to $k = 1$ by using (8). Because $x$ and $\lambda$ are determined by $x(1)$ and $U(t)$ through equations (7)-(9), equation (10) for $(k = 1 \cdots N)$ can be regarded as a single equation.

$$F\big(x(1),U(t)\big) := \frac{\partial H}{\partial u}\big(x(k),u(k),\lambda(k)\big) = 0\#(11)$$

Because $x(k)$ and $\lambda(k)$ for $(k = 1 \cdots N)$ are uniquely determined through equations (7)-(9) for the given $x(1)$ and $U(t)$, it is worth noting that $x(k)$ and $\lambda(k)$ depend on $x(1)$and $U(t)$. Hence, it is reasonable to consider the arguments of $F$ as $x(1)$and $U(t)$.

For a given $x(1)$ and $U(t)$, $F$ is not necessarily equal to zero, so $\|F\|$ is used to evaluate the optimality performance. If $\|F\| = 0$ is satisfied for the given $x(1)$ and $U(t)$, then the stationary conditions are

satisfied. Several algorithms have been developed such that $\|F\|$ can be decreased by suitably updating $U(t)$ as discussed below.

A conventional way to update $U(t)$ is to replace $U(t)$ with$U(t) + \alpha s$, known as the steepest descent method, where $s$ is the steepest descent direction and $\alpha$ is the step length satisfying the Armijo condition. For Newton's method, $s$ is given by the Hessian instead of the gradient. However, these methods are computationally expensive, and it was shown that the contraction mapping method is much faster than these methods.
Here, we suppose $F(x(1), U(t))$ can be described as

$$F\big(x(1), U(t)\big) = AU(t) + b\big(x(1), U(t)\big)\#(12)$$

Then, the updating low of $U(t)$ based on the contraction mapping (CM) method is given by

$$U(t) = A^{-1}b\big(x(1), U(t-1)\big)\#(13)$$

More detailed information about the implementation of contraction mapping method is provided in [15].

## IV. NUMERICAL SIMULATIONS

In this section, an illustrative example is provided to verify the effectiveness of the proposed method. In this simulation, we set the initial state as follows:

$$x(0) = [11,15,1.6,0,0,0,0.5,13,0]^{\mathrm{T}}$$

In the following, we provide the simulation results to verify the effectiveness of the proposed method. Figs. II and III show the time responses of state variables using nonlinear model predictive control based on the C/GMRES method. Figs. IV and V show the time responses of state variables using nonlinear model predictive control based on the CM method. It is seen that the time responses using the CM method is almost similar to the trajectory using the C/GMRES method.

Figs. VI and VII show the vehicle trajectory using the C/GMRES method and the CM method, respectively. It is seen that the controlled vehicle has successfully overtaken the front vehicle indicated by the red object. Simulation is performed on a laptop computer (CPU: Intel(R) Core (TM) i5-1135G7, Memory: 16.0 [GB], OS: Windows 11 Pro, Software: Matlab). The computational times per update (one control cycle) using the C/GMRES method and the CM method are listed in Table II. It is seen that the computational time of the CM method is smaller than that of the C/GMRES method.

| Method | Maximum time | Average time |
|---|---|---|
| C/GMRES | 0.021 | $2.1 \times 10^{-3}$ |
| CM | 0.018 | $4.7 \times 10^{-4}$ |

**Table II: The computational times**

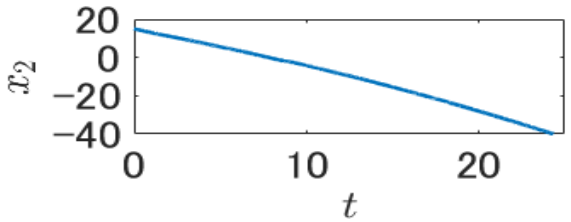


**Fig. II. Time response of$x_2(t)$of C/GMRES method**

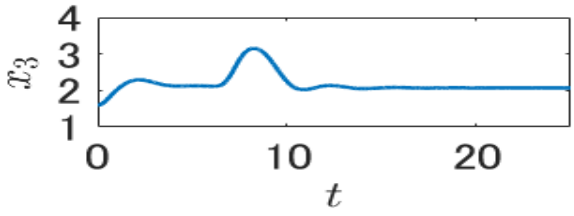


**Fig. III. Time response of$x_3(t)$of C/GMRES method**

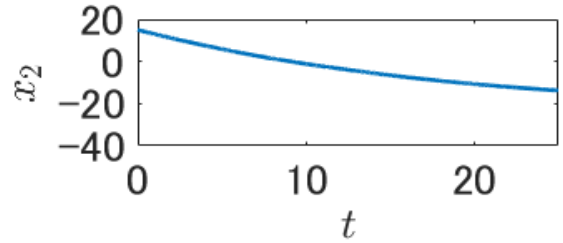


**Fig. IV. Time response of$x_2(t)$of CM method**

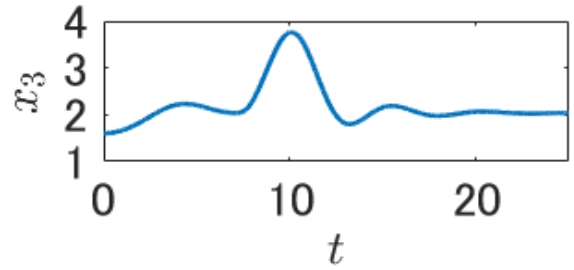


**Fig. V. Time response of$x_3(t)$of CM method**

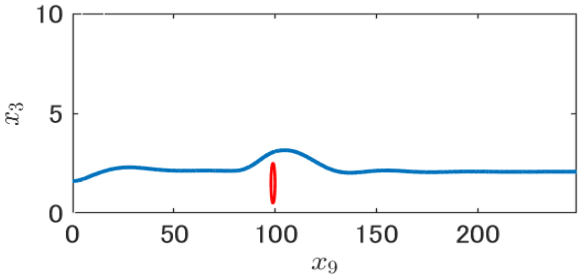


**Fig. VI. Vehicle trajectory of C/GMRES method**

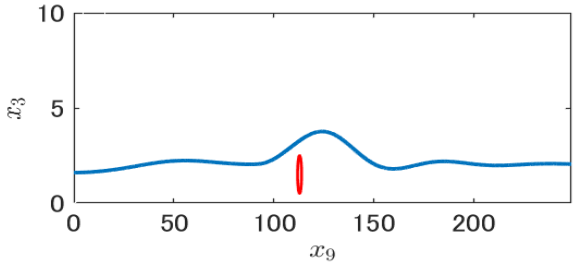


**Fig. VII. Vehicle trajectory of CM method**

## V. CONCLUSIONS

In this study, an autonomous overtaking driving system was designed based on nonlinear model predictive control (NMPC). The NMPC method previously proposed for autonomous vehicle applied the C/GMRES algorithm for numerically solving the NMPC problem. The contraction mapping (CM) algorithm is considered to be faster than the C/GMRES algorithm. The CM method is useful for solving the NMPC problem within a short sampling period. Then, the CM method was applied in this study to solving the NMPC problem of autonomous vehicle. In this study, NMPC method with a fast numerical solution algorithm for autonomous overtaking driving was established. The effectiveness of the proposed method was verified by numerical simulations.

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