A BACKWARD SEMI-LAGRANGIAN SCHEME FOR COUPLED BURGERS' EQUATION

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Abstract- This paper is intended as an application of backward semi-Lagrangian scheme combined with the error correction method to obtain numerical solutions for nonlinear coupled Burgers' equation.To linearize the nonlinear diffusion-reaction system resulted from the backward semi-Lagrangian process, we develop a type of extrapolation scheme without the damaging of temporal accuracy.Through the numerical simulations of the targeted equations, we demonstrate the proposed method has the second-order convergence rates in time and space.In addition, our numerical results are in good agreement with the analytic solution and offer better accuracy in comparison with other existing scheme.

Key word- Backward semi-Lagrangian method; Error correction method; Time dependent nonlinear partial differential equation; Coupled Burgers' equation

I. INTRODUCTION

The governing equation to be considered in this paper is the one-dimensional coupled Burgers' equation described by

$$\begin{cases} u_t - u_{xx} - 2uu_x + (uv)_x = 0, \\ v_t - v_{xx} - 2vv_x + (uv)_x = 0 \end{cases} \quad (1)$$

with initial conditions

$$u(0,x) = u_0(x), v(0,x) = v_0(x), \quad x \in [a,b]$$

and boundary conditions

$$u(t,x_L) = f_1(t), \quad u(t,x_R) = f_2(t), \quad t \in [0,T],$$

where $u_0(x)$, $v_0(x)$ and $f_1(t)$ $(i = 1,2,3,4)$ functions.The backward semi-Lagrangian method (BSLM) is one of popular strategies to solve the nonlinear partial differential equation (PDE) describing the advection process. In this paper, we apply it to solve (1) as follows: let $\pi(t)$ and$\phi(t)$arethe characteristic curves satisfying the initial value problems(IVPs)

$$\frac{d\pi(t)}{dt} = (v - 2u)\big(t,\pi(t)\big), \quad (2)$$

$$\frac{d\phi(t)}{dt} = (u - 2v)\big(t,\phi(t)\big), \quad (3)$$

respectively. Then, the equation (1) can be changed in the form

$$u_T\big(t,\pi(t)\big) = u_{xx}\big(t,\pi(t)\big) - uv_x\big(t,\pi(t)\big), \quad (4)$$

$$v_T\big(t,\varphi(t)\big) = v_{xx}\big(t,\varphi(t)\big) - vu_x\big(t,\phi(t)\big), \quad (5)$$

where the subscript Trefers to the total derivative of u and v along the characteristics $\pi(t)$ and $\phi(t)$, respectively. From the equations (4) and (5), we can see that the conventional BSLM must be modified to resolve (4) and(5) because these equations have a form of the nonlineardiffusion-reaction equation. As a modification, we construct an efficient algorithm, for which we suggest a type of extrapolation scheme without the damaging of temporalaccuracy to linearize the nonlinear advection terms of (4) and (5).

This paper is organized as following. BSLM for solving the coupled Burgers' equation is presented in Section 2. Section 3 is described ECM for solving (2) and (3). To demonstrate the accuracy and efficiency of the present method, several examples are tested in Section 4 with a numerical comparison. Finally, we provide some comments and conclusion in Section 5.

II. BACKWARD SEMI-LAGRAGIAN FINITE DIFFERENCE SCHEME

In this section, we describe BSLM with the second-order central difference formula for discretizing (4) and (5) together with an error correction scheme (ECM) for discretizing (2) and (3). Before that, we assume that the temporal and the spatial

$$0 = t_0 < t_1 < \cdots < t_M = T, \quad t_m = mh, \quad m = 0,1,\cdots,M,$$
$$a = x_0 < x_1 < \cdots < x_J = b, \quad t_i = x_0 + j\Delta x, \quad j = 0,1,\cdots,J,$$

Where $h := T/M$ is the uniform time step size, $\Delta x := (b-a)/J$is the uniform spatial grid size. For the convenience, we define some notations as follows:

$$\begin{aligned} \boldsymbol{u}^m &:= \big[u(t_m,x_0), u(t_m,x_1), \cdots u\big(t_m,x_J\big)\big]^T, \\ \boldsymbol{v}^m &:= \big[v(t_m,x_0), v(t_m,x_1), \cdots v\big(t_m,x_J\big)\big]^T, \\ \boldsymbol{U}^m &:= \big[U_0^m, U_1^m, \cdots U_J^m\big]^T, \\ \boldsymbol{V}^m &:= \big[V_0^m, V_1^m, \cdots V_J^m\big]^T, \end{aligned} \quad (6)$$

whereU_j^m and V_j^m denote the approximations of the analyticsolutions $u\big(t_m,x_j\big)$and$v\big(t_m,x_j\big)$ respectively. Additionally,we assume that the approximation vectors $\boldsymbol{U}^m$and $\boldsymbol{V}^m$arealready calculated at each grid point $x_j, (j = 0,1,\dots,J)$.

Now, let $\pi_j(t)$be a characteristic curve satisfying (2) arriving at time t_{m+1}on the grid point x_j. We apply the BDF2 and the second-order central difference formula for the total time derivative and the spatial derivative, respectively. Also, we use the following extrapolation technique

$$\frac{\partial v(t_{m+1},x_j)}{\partial x} = \left(2\frac{\partial v(t_m,x_j)}{\partial x} - \frac{\partial v(t_{m-1},x_j)}{\partial x}\right) + O(h^2) \quad (7)$$

Then, (4) can be discretized as follows:

$$\left(1 + 2\mu - \frac{2h}{3}\left(2\frac{\partial v(t_m,x_j)}{\partial x} - \frac{\partial v(t_{m-1},x_j)}{\partial x}\right)\right)u(t_{m+1},x_j). \\ -\mu\left(u(t_{m+1},x_{j-1}) + u(t_{m+1},x_{j+1})\right) \quad (8)$$
$$= \frac{4}{3}u\left(t_m,\pi_j(t_m)\right) - \frac{1}{3}u\left(t_{m-1},\pi_j(t_{m-1})\right) + O(h^3 + h\Delta x^2),$$

where $\mu = \frac{2h}{3\Delta x^2}$. Here, we notice that the departure points of the particles arriving at grid points do not typically coincide with the grid points, thus the solutions at the departure positions must be interpolated. As an interpolation scheme, we employ the cubic Lagrange interpolation (refer to the paper ([1]).

Since

$$\pi_j(t) = \pi_j(t_{m-1}) + (t - t_{m-1})(v - 2u)\left(t_{m-1},\pi_j(t_{m-1})\right) \\ + \frac{(t-t_{m-1})^2}{4h^2}\left(x_j - \pi_j(t_{m-1}) - 2h(v - 2u)\left(t_{m-1},\pi_j(t_{m-1})\right)\right) + \\ O(h^3). \quad (9)$$

evaluating (9) at time $t_m$and using the cubic Lagrange interpolation I_L, we can get an approximate value of $\pi_j(t_m)$given by

$$\pi_j(t_m) \approx \\ \frac{1}{4}(x_j + 3\pi_j(t_{m-1}) + 2h(I_L\boldsymbol{U}^{m-1} - 2I_L\boldsymbol{U}^{m-1})\left(\pi_j(t_{m-1})\right)) \quad (10)$$

To approximate $\pi_j(t_{m-1})$ we employ ECM which is discussed in the next section.

Now, substituting the approximation values of the $\pi_j(t_{m-1})$ and $\pi_j(t_m)$ (8) and dropping the unknown truncation term, we obtain the following full-discretization system

$$\boldsymbol{A}\boldsymbol{U}^{m+1} = \boldsymbol{B}^{m+1}, \quad (11)$$

where $(J-1)\times(J-1)$ tridiagonal matrix $\boldsymbol{A} := (a_{ij})$ whose $(i,j)^{th}$ entries are defined by

$$a_{ij} = \begin{cases} 1 + 2\mu & i = j \\ -\mu & |i-j| = 1 \\ 0 & otherwise \end{cases} \quad (12)$$

and

$\boldsymbol{B} = [b_1^{m+1} + \mu f_1(0), b_2^{m+1}, \cdots, b_{J-2}^{m+1}, b_{J-1}^{m+1} + \mu f_2(T)]^T$.

Here,

$$b_j^{m+1} := \frac{4}{3}I_L\boldsymbol{U}^m\left(\pi(t_m,x_j)\right) \\ -\frac{1}{3}I_L\boldsymbol{U}^{m-1}\left(\pi(t_{m-1},x_j)\right)$$

with the approximation values of $\pi(t_{m-1},x_j)$and$\pi(t_m,x_j)$.With a similar process, one can get an approximation$\boldsymbol{V}^{m+1}$.

III. ERROR CRRECTION METHOD

This section focuses on finding approximation values of the

characteristic curves $\pi_j(t)$ att_{m-1}such that

$$\begin{cases} \frac{d\pi_j(t)}{dt} = (v - 2u)\left(t,\pi_j(t)\right), & t < t_{m+1} \\ \pi_j(t_{m+1}) = x_j, \end{cases} \quad (13)$$

which is a highly nonlinear equation with self-consistent dueto the unknown slope function $v - 2u$, which are the linearcombination of the solutions $u$and $v$for (4) and (5). Mostof the process discussed below follows the ECM developed in [2,5,6].

Since $\pi_j(t)$ is the position of the trajectory of a particle,

it can be assumed that the characteristic curves $\pi_j(t)$ are

sufficiently smooth with respect to both variables, t and

x_j . Then, the Taylor's expansions of $\pi_j(t)$ about t_{m-1} and $(v-2u)\left(t_{m+1},\pi_j(t_{m+1})\right)$ about t_m give the following asymptotic formula

$$\pi_j(t) = x_j + (t - t_{m+1})(v - 2u)(t_{m+1},x_j) + O(h^2) \\ = x_j + (t - t_{m+1})(v - 2u)(t_m,x_j) + O(h^2). \quad (14)$$

From (14), we suggest a modified Euler's polygon$y_j(t)$as follows:

$$y_j(t) := x_j + (t - t_{m+1})(v - 2u)(t_m,x_j). \quad (15)$$

Let $\psi_j(t)$ be the difference between $\pi_j(t)$ and $y_j(t)$defined as

$$\psi_j(t) := \pi_j(t) - y_j(t) \quad . \quad (16)$$

After differentiating (16), combining the result with (15), one may see that$\psi_j(t)$satisfies an asymptotic first-order ordinary differential equation (ODE) given by

$$\psi_j'(t) \approx \left((v-2u)_x\left(t_n,y_j(t_m)\right)\right)\psi_j'(t) \\ +(v-2u)\left(t,y_j(t)\right) - (v-2u)(t,x_j), \quad (17)$$

where $(v-2u)_x$denotes the Jacobian of $v-2u$. By integrating both sides of equation (17) over the interval $[t_{m-1},t_{m+1}]$ and applying the mid-point integrating rule, one may get an asymptotic formula given by

$$\psi_j(t_{m-1}) \approx \left((v-2u)_x\left(t_m,y_j(t_n)\right)\right)\psi_j(t_m) \\ + (v-2u)(t,y_i(t)) \\ - (v-2u)\left(t,y_j(t)\right) \\ - (v-2u)(t,x_j). \quad (18)$$

Since $\psi_j(t_m) = \frac{1}{2}\left(\psi_j(t_{m+1}) + \psi_j(t_{m-1})\right) + O(h^2)$,theabove equation can be rewritten as

$$\left(1 + h\left((v-2u)_x\left(t_m,y_j(t_m)\right)\right)\right)\psi_j(t_{m-1}) \\ \approx 2h\left((v-2u)(t_m,x_j) - (v-2u)\left(t_m,y_j(t_m)\right)\right) \\ + O(h^3). \quad (19)$$

Also, since we assume that the approximations $\boldsymbol{U}^m = [U_0^m,\cdots,U_J^m]$and $\boldsymbol{V}^m = [V_0^m,\cdots,V_J^m]$ at time t_m are already calculated, we can approximate $y_j(t_{m-1}) = x_j - 2h(v-2u)(t_m,x_j)$ using U_j^m and

V_j^mas follows:

$$y_j^{m-1} = x_j - 2h\left(V_j^m - 2U_j^m\right). \quad (20)$$

Further, we use the fact

$$\psi_j(t_m) = \frac{1}{2}\left(\psi_j(t_{m+1}) + \psi_j(t_{m-1})\right) + O(h^2)$$

and the interpolation for $u\left(t_m, y_i(t_m)\right)$and $v\left(t_m, y_i(t_m)\right)$. Then, we can obtain the approximation of $\psi_j(t_{m-1})$ fromthe following formula

$$\left(1 + h\frac{\partial}{\partial x}\left(I_L \boldsymbol{V}^m(y_j^m) - 2I_L \boldsymbol{U}^m(y_j^m)\right)\right)\psi_j(t_{m-1}) \approx 2h\left(\left(V_j^m - I_L \boldsymbol{V}^m(y_j^m)\right) - 2\left(U_j^m - I_L \boldsymbol{U}^m(y_j^m)\right)\right). \quad (21)$$

IV. NUMERICAL EXPERIMENTS

Example

To observe the temporal and the spatial convergence orders and verify the superiority of the proposed method, wetest the coupled Burgers' equations on $[-\pi, \pi]$

$$\begin{cases} u_t - u_{xx} - 2uu_x + (uv)_x = 0, \\ v_t - v_{xx} - 2vv_x + (uv)_x = 0. \end{cases} \quad (22)$$

The boundary and the initial conditions are calculated from
the exact solution [4,7]

$$\begin{cases} u(t,x) = e^{-t}\sin(x) & x \in [-\pi, \pi], \quad t > 0, \\ v(t,x) = e^{-t}\sin(x) & x \in [-\pi, \pi], \quad t > 0. \end{cases}$$

To measure the errors, we use the maximum error $E_\infty(t)$andthe relative L_2norm error $E_2(t)$defined by

$$Err_\infty(t_k) = max_j\, |u^k(x_j - U_j^k|$$

and

$$\mathrm{Err}_2(t_k) = \left(\sum_j \left|u^k(x_j) - U_j^k\right|^2\right)^{1/2} \Bigg/ \left(\sum_j \left|u^k(x_j)\right|^2\right)^{1/2},$$

where $u^k(x_j)$is the exact solution at time$t = t_k\,(k = 1, \cdots, M)$on grid point $x_j$and $U_j^k$is its approximation foreach$0 \le j \le J$.

The temporal convergent rate for u and v are calculated byranging the temporal step size from $1/20$to $1/160$and the fixed spatial grid size $\Delta x = 2\pi/1000$at two different time levels $t = 0.1, 1.0$.The numerical results are listed in Table I and II. They clearly show that the proposed method has a numerically almost second-order temporal convergence rate.

Table 1TEMPORAL CONVERGENC RATE OF $\boldsymbol{u}$WITH$\Delta \boldsymbol{x} = \boldsymbol{2\pi/1000}$AT DIFFERENT TIMES.

h	$t = 0.1$				$t = 1.0$			
	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*
1/20	9.39e-5	-	8.73e-5	-	1.63e-4	-	3.63e-4	-
1/40	4.30e-5	1.12	4.01e-5	1.12	3.96e-5	2.04	8.89e-5	2.03
1/80	1.35e-5	1.67	1.24e-5	1.69	9.11e-6	2.12	2.14e-5	2.05
1/160	3.86e-6	1.81	3.38e-6	1.88	2.30e-6	1.99	5.58e-6	1.94

Table 2TEMPORAL CONVERGENC RATE OF $\boldsymbol{v}$WITH$\Delta \boldsymbol{x} = \boldsymbol{2\pi/1000}$AT DIFFERENT TIMES.

h	$t = 0.1$				$t = 1.0$			
	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*
1/20	1.19e-4	-	8.73e-5	-	2.13e-4	-	3.63e-4	-
1/40	5.56e-5	1.10	4.01e-5	1.12	5.13e-5	2.03	8.89e-5	2.03
1/80	1.70e-5	1.71	1.24e-5	1.69	1.19e-5	2.05	2.14e-5	2.05
1/160	4.37e-6	1.96	3.38e-6	1.88	3.63e-6	1.94	5.58e-6	1.94

To examine the spatial convergence rate of the proposed
method, the errors are calculated with the fixed temporal step size of $h = 0.0001$ by varying the spatial grid Δx from $\frac{2\pi}{10}$ to $\frac{2\pi}{160}$ at two different time levels $t = 0.1, 1.0$. The results are displayed in Table III and IV. The results show that the spatial convergence rate of the proposed method is 2.

Table 3 SPATIAL CONVERGENCE RATE OF $\boldsymbol{u}$ WITH $\boldsymbol{h = 0.0001}$ AT DIFFERENT TIEMS.

Δx	$t = 0.1$				$t = 1.0$			
	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*
$\frac{2\pi}{10}$	3.25e-3	-	3.25e-3	-	3.30e-2	-	3.30e-2	-
$\frac{2\pi}{20}$	8.19e-4	1.99	8.19e-4	1.99	8.23e-3	2.00	8.23e-3	2.00
$\frac{2\pi}{40}$	2.05e-4	2.00	2.05e-4	2.00	2.06e-3	2.00	2.06e-3	2.00
$\frac{2\pi}{80}$	5.13e-5	2.00	5.13e-5	2.00	5.14e-4	2.00	5.14e-4	2.00
$\frac{2\pi}{160}$	1.28e-5	2.00	1.28e-5	2.00	1.28e-4	2.00	1.28e-4	2.00

Table 4 SPATIAL CONVERGENCE RATE OF $\boldsymbol{v}$ WITH $\boldsymbol{h = 0.0001}$ AT DIFFERENT TIEMS.

Δx	$t = 0.1$				$t = 1.0$			
	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*	$\mathrm{Err}_\infty(t)$	*Rates*	$\mathrm{Err}_2(t)$	*Rates*
$\frac{2\pi}{10}$	3.25e-3	-	3.25e-3	-	3.30e-2	-	3.30e-2	-
$\frac{2\pi}{20}$	8.19e-4	1.99	8.19e-4	1.99	8.23e-3	2.00	8.23e-3	2.00
$\frac{2\pi}{40}$	2.05e-4	2.00	2.05e-4	2.00	2.06e-3	2.00	2.06e-3	2.00
$\frac{2\pi}{80}$	5.13e-5	2.00	5.13e-5	2.00	5.14e-4	2.00	5.14e-4	2.00
$\frac{2\pi}{160}$	1.28e-5	2.00	1.28e-5	2.00	1.28e-4	2.00	1.28e-4	2.00

Furthermore, to explore efficiency of the proposed method, we compare the proposed method (BSLM-ECM) with an implicit logarithmic finite-difference method (I-LFDM) recently developed by [7]. We calculate the maximum norm error and the relative L_2 norm error at the different time levels $t = 0.1, 0.5$and 1.0. The numerical results are displayed in Table V. One can see that BSLM-ECM is superior to I-LFDM in the sense of accuracy.

Table 5 COMPARISON OF THE ERRORS FOR EXAMPLE WITH $\boldsymbol{h = 0.001}$ FOR $\boldsymbol{u}$

t	Srivastava[7]		BSLM-ECM		Srivastava[7]		BSLM-ECM	
	J=200				J=400			
	$\mathrm{Err}_\infty(t)$	$\mathrm{Err}_2(t)$	$\mathrm{Err}_\infty(t)$	$\mathrm{Err}_2(t)$	$\mathrm{Err}_\infty(t)$	$\mathrm{Err}_2(t)$	$\mathrm{Err}_\infty(t)$	$\mathrm{Err}_2(t)$
0.1	9.02e-4	9.95e-4	7.43e-6	8.13e-6	8.95e-4	9.87e-4	1.93e-6	2.06e-6
0.5	6.04e-4	9.91e-4	2.50e-5	4.11e-5	6.01e-4	9.84e-4	6.29e-6	1.03e-5
1.0	3.65e-4	9.83e-4	3.02e-5	8.21e-5	3.64e-4	9.70e-4	7.55e-6	2.05e-5

CNCLUSION

We suggest a modification of BSLM combined withECM to obtain numerical solutions for nonlinear coupledBurgers' equation. Unlike the conventional BSL process, thecoupled Burgers' equation produces a form of the nonlinearadvection-diffusion equation in the process of BSLM. To
linearize the nonlinear advection-diffusion equation, we develop a type of extrapolation scheme without the damaging of temporal accuracy. Through the

numerical simulations of the coupled Burgers' equation, we demonstrate the proposed method has the second-order convergence rates in time and space. Also, our numerical results are in good agreement with the analytic solution and offer better accuracy in comparison with other existing scheme.

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