CONVERGENCE ANALYSIS OF BACKWARD SEMI-LAGRANGIAN SCHEME WITH BDF2 IN NON-LINEAR ADVECTION-DIFFUSION PROBLEMS

[1]DOJIN KIM, [2]SOYOONBAK, [3]PHILSU KIM

Department of Mathematics, Kyungpook National University, Daegu, South Korea
Email: kimdojin@knu.ac.kr, jiya525@knu.ac.kr, kimps@knu.ac.kr

Abstract—In this paper, we give a convergence analysis of a backward semi-Lagrangian scheme in a finite difference method for Burgers' equation. We apply an iteration free error correction method to solve characteristic curves of particles. We also adopt BDF2 and second order central difference method in time and space, respectively, for the time dependent diffusion equation along the characteristic curves.In the sense of the discrete ℓ^2-norm, we show that the proposed scheme has the convergencebehavior $O(h^2 + \Delta x^2 + \frac{\Delta x^{p+1}}{h})$, where $h, \Delta x$, and are the time, spatial grid size, and the order of interpolations, respectively.A numerical test is provided to support the validity of the theoretical estimates.

Index Terms—Burgers' equation, Error Analysis, Semi-Lagrangian Scheme, Error Correction Method, non-linear advection-diffusion equation

INTRODUCTION

It has been important issue to construct efficient and high order backward semi-Lagrangian (BSL) schemes for solving time-dependent advection-type equations for many decades. Especially, the convergence analysis of the schemes is a fundamental part of the study to achieve the objective inmethodology as well as the numerical analysis.

The goal of this paper is to give a concrete convergence analysis of the BSL scheme for non-linear advection-diffusion problems.

Let us consider the following two dimensional non-linear advection-diffusion problem

in $\Omega := \prod_{k=1,2}[x_{k,\min}, x_{k,max}] \subset R^2$, and $I := (t_0, T]$with Dirichlet boundary condition as follows:

(1){ (∂u/ ∂t (t, x) + v ⃗(t, x) · ∇u(t, x)
= ν Δ u(t, x) in I × Ω@u(t, x)
= g(t, x), on I
× ∂Ω @u(t_0, x)
= u^0 (x), in Ω,)┤

where $\vec{v}(t,\cdot)$represents the velocity of particles, $\nu$is a positive coefficient of the kinematic viscosity, $g$and $u^0$are given smooth boundary and initial conditions,respectively. If one define the material derivative of$u$ along the trace of particles by$\frac{du}{dt} = \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u$, then the governing equation (1) can be written as

$$(2)\ \frac{du}{dt}\big(t, \vec{\pi}(x,s;t)\big) = \nu \Delta u\big(t, \vec{\pi}(x,s;t)\big), t_0 \le t \le s < T,$$

Where $\vec{\pi}(x,s;t) = [\pi_1(x,s;t), \pi_2(x,s;t)]^T$ is the characteristic curve of the particle $x$for time $t \le s$and satisfies the non-linear initial value problem (IVP) given by

$$(3)\ \begin{cases} \frac{d}{dt}\vec{\pi}(x,s;t) = \vec{v}(t,s;t), \ \ t_0 \le t \le s < T, \\ \vec{\pi}(x,s;s) = x\ . \end{cases}$$

Depending on the velocity $\vec{v}$of the particles in (1), it can be generally classified to linear and non-linear advection-type equations.The overall convergence analysis of BSL schemes for the linear case are numerically ascertained through many different numerical methods in [7] and also analytically established up to the second temporal order (see, for example, [3],[6],[8], etc.).But, the convergence analysis of BSL schemes for the non-linear case are only answered with the first temporal order because of its complexity ([1],[2],[4],[5]) even though numerical results are presented for second order.The aim of the paper is to give a concrete convergence analysis for the non-linear advection-diffusion equations by solving (2) and (3). More precisely, we discretize (2) with the second order backward differentiation formula (BDF2) for the material derivative and the second order central finite difference for the diffusion term. Additionally, we apply error correction method (ECM), which is known as iteration free as well as a second order accurate algorithm ([9],[10]), to discretize the initial value problem (3). Further, we adopt general Lagrangian interpolation polynomials of degree p in order to evaluate the velocity $\vec{v}$ at non-grid points.A concrete convergence analysis for the proposed scheme is performed with three main tools which are inspired by the process of [8] as follows:i) summation by parts to estimate the diffusion term as discrete sense, ii) the discrete Poincaré inequality to estimate the discrete ℓ^2-norm of a vector function by the one of the difference of the vector function using the Dirichlet boundary condition, iii) a Gronwall type argument to obtain that the difference of error term occurred at the final step of the analysis is bounded by the one of previous steps. It turns out that the order of convergence for the proposed scheme has the form $O\left(h^2 + \Delta x^2 + \frac{\Delta x^{p+1}}{h}\right)$in the sense of the discrete ℓ^2-norm. To discuss the assertion, we restrict our

concern to the two dimensional Burgers' equation. That is, we take the velocity $\vec{v}$ as $\vec{u} = [u, u]^T$.

BACKWARD SEMI-LAGRANGIAN SCHEME

The time interval $[t_0, t_N]$ and spatial domain $\square$ are assumed to be uniformly discretized as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T, \quad t_n = nh,$$
$$x_{k,\min} = x_{k,0} < x_{k,1} < \cdots < x_{k,J_k} = x_{k,\max},$$
$$x_{k,\min} = x_{k,0} + i\Delta x_k, \quad k = 1,2,$$

$\square\square\square\square$ where $h := \frac{t_n}{N}$ is the uniform time step size and $\Delta x_k := \frac{x_{k,\max} - x_{k,\min}}{J_k}$ are uniform spatial mesh sizes in x_k directions. For $x = [x_1, x_2]$, let $(x - x_{i,j})^{k,l} := (x_1 - x_i)^k (x_2 - x_j)^l$ for grid points, $x_{i,j} := (x_{1,i}, x_{2,j}), C_{i,j} := [x_{1,i-1}, x_{1,i}] \times [x_{2,j-1}, x_{2,j}]$ be the $(i,j)^{\text{th}}$ local cell, and $\Omega := \cup C_{i,j}$. We define the discrete ℓ^2-norm for a vector function f defined on the grids in Ω by

$$||f||_2^2 = \Delta x_1 \Delta x_2 \sum_{i=0}^{J_1} \sum_{j=0}^{J_2} |f_{i,j}|^2$$

with $\mathbf{f} = [\mathbf{f}_{0,0}, \mathbf{f}_{1,0}, \cdots, \mathbf{f}_{J_1,0}, \mathbf{f}_{0,1}, \cdots, \mathbf{f}_{J_1,J_2}]^T$. For a given function $f(\mathbf{x})$ and a point $x \in C_{i,j}$, let $I_{i,j}^p f(\mathbf{x})$ be a $(i,j)^{th}$ local bi-polynomial interpolation of degree p defined by

$$I_{i,j}^p f(\mathbf{x}) = \sum_{k=0}^{p} \sum_{\ell=0}^{p} c_{k,\ell} (\mathbf{x} - \mathbf{x}_{i,j})^{k,\ell},$$

in which $c_{k,\ell}$ satisfies the interpolation conditions,

$$I_{i,j}^p f(\mathbf{x}_{k,\ell}) = f(\mathbf{x}_{k,\ell}) := f_{k,\ell}$$

in which $c_{k,\ell}$ satisfies the interpolation conditions,

$$I_{i,j}^p f(\mathbf{x}_{k,\ell}) = f(\mathbf{x}_{k,\ell}) := f_{k,\ell}$$

For all $i - \lfloor \frac{p+1}{2} \rfloor \le k \le i + \lfloor \frac{p}{2} \rfloor, j - \lfloor \frac{p+1}{2} \rfloor \le \ell \le j + \lfloor \frac{p}{2} \rfloor$, where $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n. We take interpolation nodes from the interior of the domain near the boundaries in order to have same degree when p is large. Using the local interpolation $I_{i,j}^p f(\mathbf{x})$, the function $f(\mathbf{x})$ defined on Ω can be approximated by

$$f(\mathbf{x}) \approx I^p \mathbf{f}(\mathbf{x}) := \sum_{i=1,j=1}^{J_1,J_2} 1_{C_{i,j}}(\mathbf{x}) I_{i,j}^p f(x),$$
$$\mathbf{f} = [f(x_{0,0}), f(x_{1,0}), \cdots, f(x_{J_1,J_2})]^T,$$

whose truncation error has the magnitude $O(\Delta x^{p+1})$, where $1_{C_{i,j}}$ denotes the indicator function. To discretize (3), it is required to modify the differentiablity of $I^p \mathbf{f}(\mathbf{x})$ on the edges, we define the derivatives $\tilde{D}^\beta I^p f(\cdot)$ whose restriction on the edges as the average of the left and right derivatives of the local interpolations (see [9] for detail) and $\tilde{D}^\beta I^p f(\cdot) := D^\beta I^p f(\cdot)$ in the interior of the cells, where $D^\beta u := \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$, $\beta = (\beta_1, \beta_2), \beta_k =$ integer $\ge 0 \ (k = 1,2)$ with $0 \le |\beta| = \beta_1 + \beta_2 \le 2$. For the simplicity notation, let us denote $u_{i,j}(t) := u(t, x_{i,j})$, $u_{i,j}^n := u_{i,j}(t_n)$, $U(t_n, \mathbf{x}_{i,j})$ the approximate of $u_{i,j}^n$ and $U_{i,j}^n := U(t_n, \mathbf{x}_{i,j})$ for all $0 \le n \le N$ and each grid point $x_{i,j}$ $(0 \le i \le J_1, 0 \le j \le J_2)$. We introduce the second order central difference operator for the second derivatives in the Laplace term as follows:

$$\delta^{(2,0)} u_{i,j}(t) := \frac{u_{i-1,j}(t) - 2u_{i,j}(t) + u_{i+1,j}(t)}{\Delta x_1^2},$$
$$\delta^{(0,2)} u_{i,j}(t) := \frac{u_{i,j-1}(t) - 2u_{i,j}(t) + u_{i,j+1}(t)}{\Delta x_2^2}. \tag{4}$$

Now we are ready to introduce a discretization scheme for the equations (2) and (3). By applying BDF2 for the material derivative and the second order central difference operator for the Laplace term in equation (2), respectively, one can get the following discretization system for the equation (2):

$$U_{i,j}^{n+1} - \frac{2}{3} h \nu (\delta^{(2,0)} + \delta^{(0,2)}) U_{i,j}^{n+1} = \frac{4}{3} I^p U^n(\vec{\pi}_{i,j}^{n+1,n}) - \frac{1}{3} I^p U^{n-1}(\vec{\pi}_{i,j}^{n+1,n-1})$$

for $1 \le i \le J_1 - 1, 1 \le j \le J_2 - 1$, where $\vec{\pi}_{i,j}^{n+1,n-1}$ and $\vec{\pi}_{i,j}^{n+1,n}$ are the approximation of $\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_{n-1})$ and $\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_n)$, respectively, which is discussed in the below.

Remark 2.1 *It can be shown that the equation (5) is solvable by an eigenvalue analysis for the discrete system.*

Now, we present a review for an error correction strategy recently developed by [9, 10] to find the approximate values $\vec{\pi}_{i,j}^{n-1}$ and $\vec{\pi}_{i,j}^{n}$. Let $\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t)$ be the characteristic curve satisfying the following non-linear IVP described as

$$\begin{cases} \frac{d\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t)}{dt} = \vec{u}(t, \vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t)), & t \in [t_{n-1}, t_{n+1}), \\ \vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_{n+1}) = \mathbf{x}_{i,j}, \end{cases} \tag{6}$$

where $\vec{u}(t, \cdot) = [u(t, \cdot) \ v(t, \cdot)]^T$ is the velocity field and $u(t, \cdot)$ is the solution of the model equation (1).

Based on the Euler polygon $\mathbf{y}_{i,j}(t)$ given by

(7) $\mathbf{y}_{i,j}(t) := \mathbf{x}_{i,j} + (t - t_{n+1})\vec{U}(t_n, \mathbf{x}_{i,j})$,

we apply the error correction method ([9, 10]) with the interpolation polynomial $\vec{I}\vec{U}^n$ to get the explicit formulas for the approximates $\vec{\pi}_{i,j}^{n+1,n-1}$ and $\vec{\pi}_{i,j}^{n+1,n}$ of $\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_{n-1})$ and $\vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_n)$, respectively, given by

$$\vec{\pi}_{i,j}^{n+1,n-1} := \mathbf{x}_{i,j} - 2h\vec{U}_{i,j}^n + 2h\left(I + h\frac{\partial \vec{I}\vec{U}^n}{\partial \mathbf{x}}(\mathbf{y}_{i,j}^n)\right)^{-1} \times \left(\vec{U}_{i,j}^n - \vec{I}\vec{U}^n(\mathbf{y}_{i,j}^n)\right)$$

and

$$\vec{\pi}_{i,j}^{n+1,n} = \frac{1}{4}\left(\mathbf{x}_{i,j} + 3\vec{\pi}_{i,j}^{n+1,n-1} + 2h\vec{I}\vec{U}^{n-1}(\vec{\pi}_{i,j}^{n+1,n-1})\right),$$

wher

$$\frac{\partial \vec{I}\vec{U}^n}{\partial \mathbf{x}}(\mathbf{y}_{i,j}^n) := \begin{bmatrix} \frac{\partial}{\partial x_1}\vec{I}U^n(\mathbf{y}_{i,j}^n) & \frac{\partial}{\partial x_2}\vec{I}U^n(\mathbf{y}_{i,j}^n) \\ \frac{\partial}{\partial x_1}\vec{I}U^n(\mathbf{y}_{i,j}^n) & \frac{\partial}{\partial x_2}\vec{I}U^n(\mathbf{y}_{i,j}^n) \end{bmatrix}.$$

Remark 2.2 The truncation errors

$$e_{i,j}^{n+1,k} := \vec{\pi}(\mathbf{x}_{i,j}, t_{n+1}; t_k) - \vec{\pi}_{i,j}^{n+1,k}, k = n-1, n$$

have the following asymptotic behavior

$$e_{i,j}^{n+1,k} = O\left(h^3 + \frac{h^3 + h\Delta x}{\Delta x}E_{i,j}^k + h\Delta x^{p+1}\right),$$

provided $h/\Delta x = O(1)$.

I. Convergence Analysis

The aim of this section is to provide the discrete ℓ^2-norm error estimate of the proposed scheme (5) using the main three tools as follows: discrete version of integration by parts, summation by parts, the discrete Poincaré inequality, a Gronwall type argument.

For the convenience, let $E_{i,j}^n$ be the error defined by $E_{i,j}^n := u_{i,j}^n - U_{i,j}^n$ and $E^n = [E_{0,0}^n, \cdots, E_{J_1,J_2}^n]^T$. Also, we denote

$$\begin{aligned} \Delta_t^- E_{i,j}^{n+1} &:= E_{i,j}^{n+1} - E_{i,j}^n, \\ \delta^{(1,0)}E_{i+1,j}^n &:= (E_{i+1,j}^n - E_{i,j}^n)/\Delta x_1, \\ \delta^{(0,1)}E_{i,j+1}^n &:= (E_{i,j+1}^n - E_{i,j}^n)/\Delta x_2, \end{aligned}$$

and

$$\begin{aligned} \Delta_t^- E^{n+1} &:= [\Delta_t^- E_{0,0}^{n+1}, \cdots, \Delta_t^- E_{J_1,J_2}^{n+1}]^T, \\ \delta^{(1,0)}E^n &:= [\delta^{(1,0)}E_{1,1}^n, \cdots, \delta^{(1,0)}E_{J_1,J_2}^n]^T, \\ \delta^{(0,1)}E^n &:= [\delta^{(0,1)}E_{1,1}^n, \cdots, \delta^{(0,1)}E_{J_1,J_2}^n]^T. \end{aligned}$$

From the diffusion equation (2) and the approximation scheme (5) and using Young's inequality $(ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \varepsilon > 0)$ together with tedious manipulations, one can get the following inequality for the error $E_{i,j}^n$

(8)

$$\begin{aligned} &(\Delta_t^- E_{i,j}^{n+1})^2 - \frac{2}{3}h\nu(\delta^{(2,0)} + \delta^{(0,2)})E_{i,j}^{n+1}\Delta_t^- E_{i,j}^{n+1} \le \\ &\frac{4\varepsilon}{6}(\vec{I}E^n(\pi_{i,j}^{n+1,n}) - E_j^n)^2 + \frac{\varepsilon}{6}(\vec{I}E^{n-1}(\pi_{i,j}^{n+1,n-1}) - E_{i,j}^{n-1})^2 \\ &\frac{1}{6}\left(\Delta_t E_{i,j}^n\right)^2 + (\frac{1}{6} + \frac{4}{3\varepsilon})(\Delta_t E_{i,j}^{n+1})^2 + (O(h, \Delta x)_{i,j}^{n+1})^2. \end{aligned}$$

To estimate the discrete Laplace term in the left hands side of (8), we use the discrete version of integration by parts

$$\sum_{j=m}^{M}(P_{j+1} - P_j)Q_j = -\sum_{j=m}^{M}P_{j+1}(Q_{j+1} - Q_j) + P_{M+1}Q_{M+1} - P_m Q_m$$

in order to obtain

$$\begin{aligned} &\sum_{i=1}^{J_1-1}\sum_{j=1}^{J_2-1}(\delta^{(2,0)} + \delta^{(0,2)})E_{i,j}^{n+1}(\Delta_t^- E_{i,j}^{n+1}) \\ &= -\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\delta^{(1,0)}E_{i+1,j}^{n+1}\left(\Delta_t^-\delta^{(1,0)}E_{i+1,j}^{n+1}\right) \\ &\quad -\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\delta^{(0,1)}E_{i,j+1}^{n+1}\left(\Delta_t^-\delta^{(0,1)}E_{i,j+1}^{n+1}\right). \end{aligned} \tag{9}$$

The difference of E^n between grid points and non-grid points are estimated as follows. Let $\vec{\pi}_{i,j}^{n+1}(t)$ be the approximation of characteristic curve $\vec{\pi}(\mathbf{x}_{i,j}, t^{n+1}; t)$ from t_{n-1} to t_{n+1} such that $\vec{\pi}_{i,j}^{n+1}(t_{n+m}) = \vec{\pi}_{i,j}^{n+1,n+m}$ $(m = -1, 0, 1)$ and $\vec{\pi}_{i,j}^{n+1,n+1} := \mathbf{x}_{i,j}$. Suppose that $\vec{\pi}_{i,j}^{n+1}(t) \in C_{i,j}$ and $\left|\frac{d\vec{\pi}_{i,j}^{n+1}(t)}{dt}\right| \le M$ for all $n = 1, \cdots, N$, $1 \le i \le J_1 - 1, 1 \le j \le J_2 - 1$, and for some $M > 0$. Then there exist constants $c_1, c_2 > 0$ depending on the interpolation $\vec{I}$ of the degree p such that

$$\begin{aligned} |\vec{I}E^k(\vec{\pi}_{i,j}^{n+1,k}) - E_{i,j}^k|^2 &\le c_1 h^2\left(|\delta^{(1,0)}E_{i,j}^k|^2 + |\delta^{(0,1)}E_{i,j}^k|^2\right) \\ &\quad + O(\Delta x^{2(p+1)}), k = n-1, n, \end{aligned}$$

for all $n = 1, \cdots, N$. By the definition of $\vec{I}$, $E_{i,j}^{n-1}$, and the assumption $|\frac{d\vec{\pi}_{i,j}^{n+1}}{dt}| \le M$ one can have the conclusion.The result of remark 2 provides the local truncation error in (8) given by

$$O(h, \Delta x)_{i,j}^{n+1} := O\left(h^3 + h\Delta x^2 + \Delta x^{p+1} + hE_{i,j}^n\right)$$

for $1 \le i \le J_1, 1 \le j \le J_2 - 1$.

Remark 3.1 *Using the property of the telescope summation formula with the boundary condition, we*

obtain $\left|E^n_{m,l}\right|^2 = \left(\Delta x_1 \sum_{i=1}^{m} \delta^{(1,0)} E^n_{i,l}\right)^2 \le J_1 \Delta x_1^2 \sum_{i=1}^{m} \left|\delta^{(1,0)} E^n_{i,l}\right|^2$

$\le J_1 \Delta x_1^2 \sum_{i=1}^{J_1} \left|\delta^{(1,0)} E^n_{i,l}\right|^2$.

We multiply by $\Delta x_1 \Delta x_2$ and sum $1 \le m \le J_1, 1 \le l \le J_2$ in the above equation to obtain

$$\| E^n \|_2^2 \le J_1^2 \Delta x_1^2 \| \delta^{(1,0)} E^n \|_2^2 .$$

Similarly, $\| E^n \|_2^2$ is also bounded by $\| \delta^{(0,1)} E^n \|_2^2$ and one may get

(10)

$$\| E^n \|_2^2 \le 2C_J^2 \left(\| \delta^{(1,0)} E^n \|_2^2 + \| \delta^{(0,1)} E^n \|_2^2 \right)$$

for some $C_J = \max \{ | x_{k,\max} - x_{k,\min} | \}$ using $J_k \Delta x_k = | x_{k,\max} - x_{k,\min} |$. Note that one can interpret (10) as the discrete Poincaré inequality and we introduce a notation $\delta^{(1)} E^n := \sqrt{\| \delta^{(1,0)} E^n \|_2^2 + \| \delta^{(0,1)} E^n \|_2^2}$ for convenience.

Applying the result (9), the property (10) and Young's inequality in the summation of equation (8) for all i, j, n and combining with the results of lemma 3 and remark 2, we obtain

$$\frac{1}{6} \| \Delta_t^- E^N \|_2^2 + \frac{1}{2} h \nu | \delta^{(1)} E^N |^2 \le \frac{1}{6} \| \Delta_t^- E^1 \|_2^2 + \frac{1}{2} h \nu | \delta^{(1)} E^1 |^2$$

(11)

$$+ Ch^2 \sum_{n=1}^{N-1} | \delta^{(1)} E^n |^2 + O\left(h^5 + \frac{\Delta x^{2(p+1)}}{h} + h \Delta x^4 \right).$$

Finally, one can have the following convergence result based on inequality (11).

Theorem 3. Suppose that u and $U^n_{i,j}$ satisfy the equations (1) and (5), respectively, for all $n = 1, \cdots, N-1$. If $\| \Delta_t^- E^1 \|_2$ and $| \delta^{(1)} E^1 |$ are bounded, then it is satisfied that

$$\| E^N \|_2 \le C'_N \frac{1}{\sqrt{\nu}} \left(\frac{1}{\sqrt{6h}} \| \Delta_t^- E^1 \|_2 + | \delta^{(1)} E^1 | + O\left(h^2 + \frac{\Delta x^{p+1}}{h} + \Delta x^2 \right) \right)$$

for some constant $C'_N > 0$.

Proof. Applying the fact (10) in the equation (11) with the discrete Gronwall's inequality, we can prove the assertion.

NUMERICAL EXPERIMENT

To verify the convergence analysis in Theorem 3, we carry out numerical simulations using the simple numerical tests with the two dimensional unsteady Burgers' equation. To measure the computational errors of the proposed scheme, we use the discrete ℓ^2-norm error $Err_{l_2}(t)$ defined as

$$Err_{l_2}(t_n) = \sqrt{\Delta x_1 \Delta x_2 \sum_{i,j} (err^n_{i,j})^2},$$

where $err^n_{i,j} = | u^n_{i,j} - U^n_{i,j} |$, $u^n_{i,j}$ is exact solution and $U^n_{i,j}$ is its approximation at time $t = t_n$ on each grid point $\mathbf{x}_{i,j}$.

Example. We consider the two dimensional unsteady Burgers' equation described as for $\mathbf{x} \in [-3, 3]^2, t \in (0, 1]$,

$$u_t(t, \mathbf{x}) + \vec{u}(t, \mathbf{x}) \cdot \nabla u(t, \mathbf{x}) = \nu \Delta u(t, \mathbf{x}).$$

The initial condition is given as a 2D Gaussian function as follows:

$$u(\mathbf{x}) = A \exp\left(-\left(\frac{(x_1 - c_{x_1})^2}{2\sigma_{x_1}^2} + \frac{(x_2 - c_{x_2})^2}{2\sigma_{x_2}^2} \right) \right),$$

where the coefficient A is the amplitude, (c_{x_1}, c_{x_2}) is the center and $\sigma_{x_1}, \sigma_{x_2}$ are the x_1 and x_2 spreads of the blob. We test with $A = 1, c_{x_1} = c_{x_2} = 0$ and $\sigma_{x_1} = \sigma_{x_2} = 1$.

We examine the temporal convergence rate of the proposed method with two viscosity coefficients $\nu = 0.1$ and 0.01. For the spatial step size $\Delta x_1 = \Delta x_2 = 1/300$ and the temporal time steps h from $1/5$ to $1/80$, we measure Err_{l_2} using the quadratic Lagrange interpolation and list the results in Table I. The table shows that the proposed method has almost second-order temporal convergence rate.

Table 1: Temporal convergence rate at $T = 1.0$ with $\Delta x_1 = \Delta x_2 = 1/300$.

h	$\nu = 0.1$		$\nu = 0.01$	
	$Err_{l_2}(T)$	Rate	$Err_{l_2}(T)$	Rate
1/5	2.39e-2	-	3.59e-2	-
1/10	5.94e-3	2.01	9.11e-3	1.98
1/20	1.48e-3	2.00	2.26e-3	2.01
1/40	3.85e-4	1.95	5.43e-4	2.06
1/80	1.06e-4	1.85	1.45e-4	1.90

In addition, to assess the spatial convergence rate of the proposed method, Err_{l_2} is calculated using the quadratic Lagrange interpolation with the fixed temporal step size $h = 0.001$, and with two viscosity coefficients $\nu = 0.1$ and 0.01. Especially, we take the spatial grid sizes

$\Delta x_1 = \Delta x_2$ varying from $1/50$ to $1/800$ at $T = 1$, so that the error dominates the $O(\Delta x^2)$ term, where $\Delta x = \max\{\Delta x_1, \Delta x_2\}$. The results are displayed in Table II and indicate that the rate of spatial convergence is 2.

Table 2: Spatial convergence rate at $T = 1.0$ with $h = 0.001$.

	$\nu = 0.1$		$\nu = 0.01$	
$\Delta x_1 = \Delta x_2$	$Err_{l_2}(T)$	Rate	$Err_{l_2}(T)$	Rate
1/50	3.91e+0	-	7.02e+0	-
1/100	9.25e-1	2.08	1.76e+0	2.00
1/200	2.18e-1	2.08	4.06e-1	2.11
1/400	5.08e-2	2.10	9.16e-2	2.15
1/800	1.11e-2	2.19	1.96e-2	2.22

REFERENCES

[1]. A. Allievi, R. Bermejo, Finite element modified method of characteristics for the Navier-Stokes equations, Int. J. Numer. Methods Fluids, 32(4), 439-463, (2000).

[2]. A. Priestley, Exact projections and the Lagrange-Galerkin method: a realistic alternative to quadrature, J. Comput. Phys., 112(2), 316-333, (1994).

[3]. D. Xiu, G. E. Karniadakis, A Semi-Lagrangian High-Order Method for Navier-Stokes Equations, Journal of Computational Physics, 172(2), 658-684, (2001).

[4]. E. Süli, E., Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations, Numer. Math., 53(4), 459-483, (1988).

[5]. J. Douglas, Jr., T. F. Russell, Numerical Methods for Convection-Dominated Diffusion Problems Based on Combining the Method of Characteristics with Finite Element or Finite Difference Procedures, SIAM J. Numer. Anal., 19(5), 871-885, (1982).

[6]. J.-M. Qiu,C.-. Shu, Conservative high order semi-Lagrangian finite difference WENO methods for advection in incompressible flow, Journal of Computational Physics, 230, 863-889, (2011).

[7]. M. Falcone, R. Ferretti, Convergence Analysis for a class of High-order Semi-Lagrangian Advection Schemes, SIAM J. Numere. Anal., 35(3), 909-940, (1998).

[8]. P. G. del Sastre, R. Bermejo, Error Analysis for hp-FEM Semi-Lagrangian Second Order BDF Method for Convection-Dominated Diffusion Problems, J. Sci. Comput., 49, 211-237, (2011).

[9]. X. Piao, S. Kim, P. Kim, An Iteration Free Backward Semi-Lagrangian Scheme for Guiding Center Problems, SIAM Journal on Numerical Analysis, 53(1), 196-643, (2015).

[10]. X Piao, S. Bu, S. Bak, P. Kim, An iteration free backward Semi-Lagrangian Scheme for solving incompressible Navier-Stokes Equations, Journal of Computational Physics, 283, 189-204, (2015).

★★★