INFINITESIMALS AND INFINITIES AND MATHEMATICS AND PHYSICS OF THE FUTURE

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Abstract - This brief report is an effort to comment on the possibility of integration of various infinitesimals and infinites definitions in physics, where certain theories such as String Theory may have a related origin. It is an effort to see the limiting process as numerical quantity or at least with numerical properties. Such a concept may be considered as an underlying mathematical tool in physics.

Keywords - Infintesimals, Infinities, Hyperreal, Levi-Civita

I. INTRODUCTION

It is suggested that the use of infinitesimal quantities in a mathematical apparatus of both quantum theory and gravity is incorrect, despite the fact that both these theories give the results correlating well with the experiment [1]. Infinitesimals are a basic ingredient in calculus as developed by Leibniz, and they are related to the law of continuity and the transcendental law of homogeneity. An infinitesimal object is an object that is smaller than any feasible measurement, but not zero in size—or, so small that it cannot be distinguished from zero by any available means. Infintesimal means infinitely small, smaller than any standard real number. Infinitesimals are often compared to other infinitesimals of similar size, as in examining the derivative of a function. An infinite number of infinitesimals are summed to calculate an integral. On the other hand, the mathematical concept of infinity numbers and infinity is based on the old philosophical concept, in particular the introduction of infinitely many different sizes of infinite sets. Among the axioms of Zermelo–Fraenkel set theory, on which most of modern mathematics can be developed, is the axiom of infinity, which guarantees the existence of infinite sets [2].

II. HISTORY OF INFINITESIMALS

The concept of infinitesimals was originally introduced around 1670 by either Nicolaus Mercator or Gottfried Wilhelm Leibniz [3]. Archimedes used what eventually came to be known as the method of indivisibles in his work The Method of Mechanical Theorems to find areas of regions and volumes of solids [4] and he also solved the same problem using the method of exhaustion. Nicholas of Cusa in the 15th century and Johannes Kepler in the 17th century studied these ideas, in relation to the calculation of area of a circle by representing the latter as an infinite-sided polygon. Simon Stevin's work on decimal representation of all numbers in the 16th century prepared the ground for the real continuum. Bonaventura Cavalieri's method of indivisibles led to an extension of the results of the classical authors. John Wallis exploited an infinitesimal denoted $1/\infty$ in area calculations.

Leonhard Euler and Joseph-Louis Lagrange used infinitesimals in the 18th century. Augustin-Louis Cauchy exploited infinitesimals both in defining continuity in his Coursd'Analyse, and in defining an early form of a Dirac delta function. Paul du Bois-Reymond wrote a series of papers on infinitesimal-enriched continua based on growth rates of functions. Du Bois-Reymond's work inspired both Émile Borel and ThoralfSkolem. Borel explicitly linked du Bois-Reymond's work to Cauchy's work on rates of growth of infinitesimals. Skolem developed the first non-standard models of arithmetic in 1934. A mathematical implementation of both the law of continuity and infinitesimals was achieved by Abraham Robinson in 1961, who developed nonstandard analysis based on earlier work by Edwin Hewitt in 1948 and Jerzy Łoś in 1955. The hyperreals implement an infinitesimal-enriched continuum and the transfer principle implements Leibniz's law of continuity. The standard part function implements Fermat's adequality (a technique developed by Pierre de Fermat to calculate maxima and minima of functions, tangents to curves, area, center of mass, least action, and other problems in calculus which he borrowed from Diophantus).

Today, the various infinitesimals and infinites found such as in Levi Civita forms, hyperreal and surreal numbers, as well as uncertainty principles in Quantum Mechanics and strings in String Theory may have a related origin, a limiting process to be seen as a numerical quantity or at least with numerical properties. Such a concept may be considered as an underlying mathematical tool in physics. it is suggested that the use of infinitesimal quantities in a mathematical apparatus of both quantum theory and

gravity is incorrect, despite the fact that both these theories give the results correlating well with the experiment [1].

III. EXAMPLES IN MATHEMATICS

In mathematics, the Levi-Civita field, named after Tullio Levi-Civita, is a non-Archimedean ordered field; i.e., a system of numbers containing infinite and infinitesimal quantities. Each member a can be constructed as a formal series of the form

$$a = \sum_{q \in Q} a_q \varepsilon^q$$

where a_q is the set of rational numbers, and ε^q is to be interpreted as a positive infinitesimal. The support of a, i.e., the set of indices of the nonvanishing coefficients$\{q \in Q : a_q \neq 0\}$, must be a left-finite set: for any member of$Q$, there are only finitely many members of the set less than it; this restriction is necessary in order to make multiplication and division well defined and unique. The ordering is defined according to the dictionary ordering of the list of coefficients, which is equivalent to the assumption that ε is an infinitesimal. The real numbers are embedded in this field as series in which all of the coefficients vanish except a_0. The system of hyperreal numbers is a way of treating infinite and infinitesimal (infinitely small but non-zero) quantities. The hyperreals, or nonstandard reals, *R, are an extension of the real numbers R that contains numbers greater than anything of the form

$$1 + 1 + 1 + \cdots + 1$$

(for any finite number of terms). Such numbers are infinite, and their reciprocals are infinitesimals. In the 1960s, Abraham Robinson proved that the hyperreals were logically consistent if and only if the reals were. The application of hyperreal numbers and in particular the transfer principle to problems of analysis is called nonstandard analysis. One can define the derivative and integral in a directly, without the use of logical quantifiers. Thus, the derivative of f(x) becomes

$$f'(x) = st\left(\frac{f(x+h) - f(x)}{h}\right)$$

where st(·) denotes the standard part function, which "rounds off" each finite hyperreal to the nearest real. Similarly, the integral is defined as the standard part of a suitable infinite sum.The surreal number system is a totally ordered proper class containing the real numbers as well as infinite and infinitesimal numbers, respectively larger or smaller in absolute value than any positive real number.

The surreals share many properties with the reals, including the usual arithmetic operations (addition, subtraction, multiplication, and division); as such, they form an ordered field. In von Neumann–Bernays–Gödel set theory (a conservative extension of Zermelo–Fraenkel-Choice set theory that introduces the notion of class, which is a collection of sets defined by a formula whose quantifiers range only over sets), the surreal numbers are a universal ordered field in the sense that all other ordered fields, such as the rationals, the reals, the rational functions, the Levi-Civita field, the superreal numbers, and the hyperreal numbers, can be realized as subfields of the surreals [5].

The surreals also contain all transfinite ordinal numbers; the arithmetic on them is given by the natural operations. It has also been shown (in von Neumann–Bernays–Gödel set theory) that the maximal class hyperreal field is isomorphic to the maximal class surreal field.

In the Conway construction [6], the surreal numbers are constructed in stages, along with an ordering ≤ such that for any two surreal numbers a and b, a ≤ b or b ≤ a. (Both may hold, in which case a and b are equivalent and denote the same number.) Each number is formed from an ordered pair of subsets of numbers already constructed: given subsets L and R of numbers such that all the members of L are strictly less than all the members of R, then the pair { L | R } represents a number intermediate in value between all the members of L and all the members of R. There are also representations for infinities and infinitesimals, like:

$$\{0,1,2,3\ldots]\} = \omega$$
$$\left\{0]1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\ldots\right\} = \varepsilon$$

IV. EXAMPLES IN PHYSICS

In quantum mechanics, everything we could possibly find out about a particle is contained in the state of the particle or system described by a wave function:

$$\psi(x,t)$$

The wave function is a solution of Schrödinger's equation:

$$-\frac{h^2}{2m}\nabla^2\psi(x,t) + V\psi(x,t) = ih\frac{\partial\psi(x,t)}{\partial t}$$

The wave function itself is not a real physical wave, rather it is a probability amplitude whose modulus squared gives the probability that the particle or system is found in a given state. There exists an uncertainty principle that puts constraints on the precision with which certain quantities can be known [7]:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

In quantum field theory, space-time is filled with fields that act as operators. A given field can be Fourier expanded as

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\vec{x})} + \varphi^*(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\vec{x})}\right]$$

We then express the fields in terms of creation and annihilation operators by making the transition

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\vec{x})} + \hat{a}^+(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\vec{x})}\right]$$

The field then creates and destroys particles that are the quanta of the given field. Particle interactions take place at specific, individual points in space-time. This is important because it means that particle interactions take place over zero distance. Particles in quantum field theory are point particles represented mathematically as located at a single point [7]. This type of procedure works well in the electromagnetic, weak, and strong interactions.

However, the overall procedure has some big problems and they cannot be dealt with when gravity is involved. The problem comes down to the fact that interactions occur at a single space-time point. This leads to infinite results in calculations (aptly called infinities). Technically speaking, the calculation of a given amplitude which includes all virtual processes involves an integral over all possible values of momentum. This can be described by a loop integral that can be written in the form:

$$I \sim \int p^{4J-8} d^D p$$

Here p is momentum, J is the spin of the particle, and D, which is seen in the integration measure, is the dimension of space-time. if the momentum p → infinity but 4J+D-8>0 the integral diverges [7]. This leads to infinities in calculations.

Now if I → infinity but does so slowly, then a mathematical technique called renormalization can be used to get finite results from calculations. Such is the case when working with established theories like quantum electrodynamics. String theory gets rid of this problem by getting rid of particle interactions that occur at a single point. In the uncertainty principle

$$\Delta x \Delta p \sim \hbar$$

if the momentum blow up, that is, $\Delta p \to \infty$, this implies that $\Delta x \to 0$. That is, large (infinite) momentum means small (zero) distance. Or put another way, point like interactions (zero distance) imply infinite momentum. This leads to divergent loop integrals, and infinities in calculations. In string theory, particles are replaced by strings, spreading out interactions over space-time so that infinities don't result [7].

V. CONCLUSION

Both in mathematics and physics we see an effort to "numericize" infinitesimals and infinites. This probably originates from the necessity to understand limiting and approximating processes such as the limit. Perhaps, in the near future, the use of systems such as the surreal numbers in physics may resolve known issues in the existing theories and, moreover, the use merging of mathematics and physics may allow concepts such as uncertainty, to shed light to the ideas of infinities and infinitesimals in mathematics.

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