ON GENERALIZATION OF COMMUTATIVE SEMIGROUP RINGS

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Abstract- Left almost semigroups (LA-semigroups) or Abel-Grassmann's groupoids (AG-groupoids) have been studied by several authors and this motivated to extend these concepts to Left Almost ring (LA-ring), which carries attraction due to its structural formation. In this paper we generalize the structure of commutative semigroup ring (ring of semigroup S over ring R represented as $R[X;S]$ to a non-associative LA-ring of LA-semigroup S over LA-ring R represented as $R[X^s; s \in S]$, consisting of finitely nonzero functions. Nevertheless it also possesses associative ring structures. Furthermore we also discuss the LA-ring homomorphisms.

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INTRODUCTION

Following M. Kazim and M. Naseerudin [3], a groupoid $(S,*)$ is called a left almost-semigroup (LA-Semigroup) or Abel-Grassmann'sgroupoid (AG-groupoid) (see [2]), if it satisfies the left invertive law; $(a*b)*c=(c*b)*a$ for all $a,b,c \in S$. By [3], every LA-semigroup $(S,*)$ is medial that is $(a*b)*(c*d)=(a*c)*(b*d)$ for all $a,b,c,d \in S$. Whereas an LA-semigroup $(S,*)$ is said to be weak associative LA-semigroup if $(a*b)*c=b*(a*c)$ or $(a*b)*c=b*(c*a)$ for all $a,b,c \in S$. Q. Mushtaq and M. S. Kamran extended the notion of LA-semigroup to left almost group (LA-group) (see [6]). A groupoid $(G,*)$ is called a left almost group (LA-group), ifthere exists left identity $e \in G$ such that $e*a=a$ for all $a \in G$, for $a \in G$ there exists $b \in G$ such that $b*a=e$ and invertive law holds in G.

In [10], T. Shah and I. Rehman have discussed LA-rings of finitely nonzero functions. By a left almost ring (LA-ring), we mean a nonempty set R with at least two elements such that $(R,+)$ is an LA-group and $(R,\cdot)$ is an LA-semigroup and both left and right distributive laws hold. For example, from a commutative ring $(R,+,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for $a,b,c \in R$, $a \oplus b = b-a$ and $a \cdot b$ is same as in the ring.

Furthermore in [10], T. Shah and I. Rehman have introduced the notion of LA-module, a non abelian non associative structure, over an LA-ring. In continuation, T. Shah and M. Raees [9], have generalized some useful results analogous to module theory for LA-modules.

The study [10], addresses the structure of left almost rings (LA-rings) of finitely nonzero functions from commutative semigroup S to LA-ring R, which in fact generalizes various analogous of corresponding parts of commutative semigroup rings. In the continuation of [10], in this study commutative semigroup S is taken as an LA-semigroup and almost all the established results in [10] stand as a particular case.

THE CONSTRUCTION

In this section we construct an LA-ring of finitely non-zero functions from LA-semigroup to LA-ring and also we discuss the concepts of degree and order of LA-ring.

Let $(R,+,\cdot)$ be an LA-ring with left identity and S be an LA-semigroupunder $*$. We name the set $\{f : f : S \to R$, where f are finitely nonzero $\}$ as T.

Define the binary operation + in T as $(f+g)(s)=f(s)+g(s)$.

$(T,+)$ is LA-group. Indeed, let $f, g \in T$. As $f(s), g(s) \in R$ for all $s \in S$. So $(f+g)(s)=f(s)+g(s) \in R$ and hence $f+g \in T$.

Let $f,g,h \in T$. As $f(s),g(s),h(s) \in R$ and left invertive law holds in $(R,+)$, So we have

$$(f+g)+h=(g+f)+h$$
$$(f+g)+h=(h+f)+g$$
$$(f+g)+h=(h+g)+f$$
$$(f+g)+h=f+(g+h)$$

Hence $(f+g)+h = f+(g+h)$.

Thus left invertive law holds in T. Define the map $o : S \to R$ such that $o(s) = 0$ for all $s \in S$,

$$(f + o)(s) = f(s) + o(s) = f(s) + 0 = f(s) = f(s)$$

This implies $o = 0$ f. Thus o is left additive identity in T. For every $f \in T$ there exists a function $-f : S \to R$ defined by

for all $s \in S$ and $((-f) + f)(s) = (-f(s)) + f(s) = -f(s) + f(s) = 0 = o(s)$.

This implies $(-f) + f = o$. So the left inverses exist in $(T, +)$ Hence $(T, +)$ is an LA-group.We can say $f + (-f) = 0$ as $-f(s)$ is also the right inverse of $f(s)$ in R by [6]. Now we define binary operation $\leftarrow$ in T as follows

$$f \leftarrow g(s) = \sum_{t u = s} f(t) g(u)$$

We claim that $(T, \leftarrow)$ is an LA-semigroup. As for $f(x)$ and $g(u) \in R$, where $t, u \in (S, \cdot)$ and $(R, +, \cdot)$ is LA-ring, $f \leftarrow g(s) \in R$. Since f, g are finitely nonzero on S, therefore $f \leftarrow g \in T$.

Let $f, g, h \in T$ and $s \in S$ such that

$$\begin{aligned}
((f \leftarrow g) \leftarrow h)(s) &= \sum_{t u = s} (f \leftarrow g)(t) h(u) = \sum_{t u = s} \left(\sum_{p q = t} f(p) g(q) \right) h(u) \\
&= \sum_{p q u = s} f(p) g(q) h(u) = \sum_{q u p = s} g(q) h(u) f(p) \\
&= \sum_{r p = s} \left(\sum_{q u = r} g(q) h(u) \right) f(p) = \sum_{r p = s} (g \leftarrow h)(r) f(p) \\
&= (g \leftarrow h) \leftarrow f)(s)
\end{aligned}$$

Hence $(f \leftarrow g) \leftarrow h = (g \leftarrow h) \leftarrow f$.

Thus left invertive law holds in $(T, \leftarrow)$ Now we have to verify that the binary operation $\leftarrow$ is distributive over addition. As $f(x), g(x)$ and $h(u) \in R$ and multiplication is distributive over addition in R, so

$$\begin{aligned}
(f \leftarrow (g + h))(s) &= \sum_{t u = s} f(t) (g + h)(u) = \sum_{t u = s} f(t) (g(u) + h(u)) \\
&= \sum_{t u = s} f(t) g(u) + \sum_{t u = s} f(t) h(u) = (f \leftarrow g)(s) + (f \leftarrow h)(s) \\
&= (f \leftarrow g + f \leftarrow h)(s)
\end{aligned}$$

Similarly

$$\begin{aligned}
((f + g) \leftarrow h)(s) &= \sum_{t u = s} (f + g)(t) h(u) = \sum_{t u = s} (f(t) + g(t)) h(u) \\
&= \sum_{t u = s} f(t) h(u) + \sum_{t u = s} g(t) h(u) = (f \leftarrow h)(s) + (g \leftarrow h)(s) \\
&= (f \leftarrow h + g \leftarrow h)(s)
\end{aligned}$$

Hence $(f + g) \leftarrow h = f \leftarrow h + g \leftarrow f$.

Thus $(T, +, \leftarrow)$ is an LA- ring of LA-semigroup $(S, \cdot)$ over LA-ring $(R, +, \cdot)$.

Degree and order of elements of LA-ring $R[X^s; s \in S]$

The concept of degree and order are not generally defined in semigroup rings unless we have to consider S, a totally ordered semigroup with 0 adjoined (that is ordered monoid). The structure of LA-ring $R[X^s; s \in S]$ is also not convenient for defining degree and order of an element unless $(S, \leq)$ is totally ordered. The concept of total ordering of LA-semigroups has been discussed by T. Shah, I. Rehman and A. Ali (see [8]).

Here we define support of $f = \sum_{i=1}^{n} f_i X^{s_i}$ abbreviated as $\text{Supp}(f) = \{s_i : f_i \neq 0\}$. The order and degree of f is defined as

$ord(f) = \min \text{Supp}(f)$ and

$\deg(f) = \max Supp(f)$.

Let $(\mathrm{Q}^+, \cdot)$ denote the group of all positive rational numbers. If we take $S = (\mathrm{Q}_0^+)^{ILAS}$, where ILAS abbreviates Initial LA-semigroup, which is made an LA-semigroup by defining the binary operation $*$ as

$$a * b = 0 \text{ if } a = 0 \text{ or } b = 0,$$
$$= b a^{n-1} \text{ if } a \neq 0 \text{ and } b \neq 0.$$

If R is an LA-ring with left identity, then for $f, g \in R[X^s; s \in (\mathrm{Q}_0^+, *)]$, the degree and order of

$$f(X) = a_0 + a_1 X^{\frac{a_1}{b_1}} + \ldots + a_n X^{\frac{a_n}{b_n}} \in R\left[X^s; s \in S\right]$$

with $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \ldots < \frac{a_n}{b_n}$, is defined as,

$ord(f) = 0$ if $a_0 \neq 0$ and

$\deg(f) = \frac{a_n}{b_n}$.

In a polynomial ring $R[X]$, for $f, g \in R[X]$,

$$\deg(f+g) \le \deg(f) \vee \deg(g) \text{ and}$$
$$ord(f+g) \ge ord(f) \wedge ord(g)$$

If R is an integral domain, then

$$\deg(f+g) = \deg(f) \vee \deg(g) \text{ and}$$
$$ord(f+g) = ord(f) \wedge ord(g)$$

But the following lemma shows a deviation for LA-ring $R\left[X^s; s \in S\right]$ of LA-semigroup $(Q_0^+, *) = S$ over LA-ring R.

MAIN RESULTS

In this section we generalize some results in [1]. Specifically, we show the necessary and sufficient condition for an LA-ring $R\left[X^s; s \in S\right]$ to be an LA-integral domain. We also discuss the homomorphisms of LA-rings.

Following external direct sum of semigroups as in [1], we define external direct sum of LA-semigroups $(S, *)$ and $(T, \#)$ as

$$S \times T = \{(s,t) : s \in S, t \in T\},$$

whereas the binary operation in $S \oplus T$ is defined as

$$(s_1,t_1)(s_2,t_2) = (s_1 * s_2, t_1 \# t_2) \text{ for } s_1, s_2 \in S \text{ and } t_1, t_2 \in T.$$

A mapping ϕ of an LA-ring $(R, +, \cdot)$ into an LA-ring $(R', +, \cdot)$ is called a homomorphism if $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ and for the ideal I of R, the mapping $\nu : R \to R/I$ defined as $\nu(a) = a + I$ is called the natural epimorphism of LA-ring R onto R/I. Let θ be an epimorphism of an LA-ring R to an LA-ring R', then $R/Ker\theta \cong R'$.

The following is a generalization of [10].

Theorem: *Let R be an LA-ring and S, T be LA-semigroups. Then*

$$R[X^{(s,t)}; (s,t) \in S \times T] \cong R[X^s; s \in S][X^t; t \in T],$$

where $S \oplus T$ the is external direct sum of LA-Semigroups S and T.

Theorem: *The polynomial LA-ring $R[\{Y_\lambda\}_{\lambda \in \Lambda}]$, where R is an LA-ring and $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a family of commuting indeterminates and $F = \sum_{\lambda \in \Lambda}^{w} Z_\lambda$ such that $Z_\lambda \cong Z_0$. Then $R[\{Y_\lambda\}_{\lambda \in \Lambda}]$ is isomorphic to LA-ring $R[X; F]$ of free commutative semigroup $(F, +)$ over R.*

By [7], a semigroup S is said to be m-torsion free if for any $x, y \in S$ there exists $m \ge 1$ with $x^m = y^m$, then $x = y$. We extend this for LA*-semigroup with left identity e.

Definition: *An LA*-semigroup $(S, *)$ with left identity e is said to be an M-torsion free if for all $x, y \in S$ there exists a subset M of Z^+ such that $1 \le m \in M$ with $x^m = y^m$ implies $x = y$.*

Example: *$(Q_0^+, *)$ is an LA*-semigroup with left identity 1, defined as*

$$a * b = 0 \text{ if } a = 0 \text{ or } b = 0,$$
$$= b * a \text{ if } a \neq 0 \text{ and } b \neq 0,$$

is an O-torsion free where O is the set of odd positive integers. For this, consider $m = 3$, Let $x^3 = y^3$. As $(Q_0^+, *)$ is an LA*-semigroup, so $x^2 * x = y^2 * y$. This implies $x = y$. Hence $(Q_0^+, *)$ is an O-torsion free LA*-semigroup. Similarly $(Z, *)$ an LA*-semigroup with left identity 0 defined as $a * b = b - a$, is an O-torsion free where O is a the set of odd positive integers.

The following is a generalized form of [10].

Theorem: *Let R be an LA-ring with left identity and let $(S, *)$ be an LA*-semigroup. Let $R[X^s; s \in S]$ be an LA-ring of S over R. Then $R[X^s; s \in S]$ is an LA-integral domain if and only if R is an LA-integral domain and S is an M-torsion free and cancellative.*

Theorem: *Let $\mu : R \to R_0$ be an LA-ring homomorphism. Let $A = \ker \mu$ and $\varphi : S \to S_0$ be an LA-semigroup homomorphism, where S, S_0*

are LA-semigroups with 0 *adjoined to them. Then the following statements holds;*

(1) $\mu^* : R\left[X^s; s \in S\right] \to R_0\left[X^s; s \in S\right]$ *is defined as* $\mu^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{s_i}$, *is LA-ring homomorphism such that* $\ker \mu^* = A[X^s; s \in S] = \ker \mu\left[X^s; s \in S\right]$.

μ^* *issurjective if* μ *is surjective.*

(2) $\varphi^* : R[X^s; s \in S] \to R[X^s; s \in S_0]$ *defined as* $\varphi^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n r_i X^{\varphi(s_i)}$, *is surjective and* $\ker \varphi^* = I$. *The ideal of* $R[X^s; s \in S]$ *generated by*

$\{r_a X^a - r_b X^b : a\mathcal{H}b, r \in R\}$. *is*

issurjective if τ *is surjective.*

$\tau : R[X^s; s \in S] \to R_0[X^s; s \in S_0]$ *defined as* $\tau(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{\varphi(s_i)}$, *is an LA-ring homomorphism such that* $\ker \tau = \ker \mu^* + \ker \varphi^* = A[X^s; s \in S] + I$. *Then* τ *is surjective if* μ *and* φ *are surjective.*

Definition:*An ideal* P *of an LA-ring* R *is called prime if and only if* $AB \subseteq P$ *implies that either* $A \subseteq P$ *or* $B \subseteq P$, *where* A *and* B *are ideals in* R.

The following is a generalized form of [10].

Corollary:*Let* A *be a proper ideal of LA-ring* R, *then* $A\left[X^s; s \in S\right]$ *is prime ideal in* $R\left[X^s; s \in S\right]$ *if and only if* A *is prime ideal in* R *and* S *is cancellative* M *-torsion free LA-semigroup.*

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