QUASI-CONFORMALLY FLAT SASAKIAN FINSLER STRUCTURES ON TANGENT BUNDLES

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Abstract— The quasi-conformal curvature tensor of Sasakian Finsler structures on tangent bundles is considered in order to contextualise quasi-conformally flat structures. Sasakian Finsler structures on tangent bundlesand their η-Einstein types are explained briefly. Some structure theorems of quasi-conformally flat Sasakian Finsler Structures on tangent bundlesare discussed.

Keywords— Quasi-conformal curvature tensor, quasi-conformally flat Sasakian Finsler structure, η-Einstein manifold.

I. INTRODUCTION

The notion of "quasi-conformal curvature tensor" for Sasakian Structures on tangent bundles of a Finsler manifold is defined. In order to study this, essential characteristics of these structures are given.
It is proven that a quasi-conformally flat Sasakian Finsler structure on HTM is η-Einstein. Additively the proof of the theorem 'A $(2n+1)$-dimensional$(n>2)HTM$ is quasi-conformally flat if and only if it is locally isometric to sphere$S^{2n+1}(1)$' is given.

II. MAIN RESULTS

Let M be an $m=(2n+1)$-dimensional manifold of classC^{∞}, $x=(x^1,\dots,x^m)$ are local coordinates of , T_xM is an m-dimensional tangent space at $x\in M$ and $y=y^i\frac{\partial}{\partial x^i}\in T_xM$. Then TM denotes $2m$-dimensional tangent bundle of M and$u=(x,y)\in TM$.
$F:TM\to[0,\infty[$ is a Finsler function with the following properties;

i) F is differentiable of class C^1on TM and differentiable of class C^{∞} on $TM_0=TM\backslash\{(x,0)\}$
ii) $F(x^i,\lambda y^i)=\lambda F(x^i,y^i)$, $\lambda>0$ and $(x^i,y^i)\in TM$.

Then $F^m=(M,F)$ is a Finsler manifold. The $m\times m$ Hessian matrix $g_{ij}(x,y)=\frac{1}{2}\left[\frac{\partial^2F^2}{\partial y^i\partial y^j}\right]$ is positive definite on $TM_0$and it g is called a Finsler metric tensor with g_{ij} coefficients [1].

Due to $(x^i,y^i)$are coordinates ofTM, $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\right\}$ are natural bases of T_uTM where $u=(x,y)\in TM$ Then $=X^i\frac{\partial}{\partial x^i}+X^j\frac{\partial}{\partial y^j}\in T_u$. $\pi:TM\to M$ is the bundle projection with $u=(x^1,\dots,x^m,y^1,\dots,y^m)\mapsto(x^1,\dots,x^m)$. So, the differential map $\pi_*:T_uTM\to T_{\pi(u)}M$ satisfies $X_u\mapsto\pi_*(X_u)=X_u[x^i]\frac{\partial}{\partial x^i}$. Hence $(\pi_*)=\{X\,|X\in T_uTM,\pi_*(X)=0\}=VTM$. Also, g is accepted as a Riemann metric onVTM.
TTM_0can be decomposed as$TTM_0=HTM\oplus VTM$ whereHTM and VTM are complementary distibutions of eachother forTTM_0.$HTM=\left(N_i^j(x,y)\right)$ is a non-linear connection on TM. $N_i^j=\frac{\partial N^j}{\partial y^i}$; non-linear connections are obtained by using spray coefficients $N^j=\frac{1}{4}g^{jk}\left(\frac{\partial^2F^2}{\partial y^k\partial x^h}y^h-\frac{\partial F^2}{\partial x^k}\right)$. Here g^{jk} is inverse matrix of g_{ij}. By using the linear connection ∇ on VTM, the pair (HTM,∇) is called a Finsler connection on M[2].
$\Gamma(TTM_0)$denotes an $\mathcal{F}(TM_0)$-module of vector fields on TM_0 where $\mathcal{F}(TM_0)$ is the algebra of smooth functions on TM_0.
For $X\in\Gamma\left(TTM_{0|u|}\right)$, if $X=X^i\frac{\partial}{\partial x^i}+X^j\frac{\partial}{\partial y^j}=X^i\frac{\partial}{\partial x^i}-N_i^j(x,y)X^i\frac{\partial}{\partial y^j}+N_i^j(x,y)X^i\frac{\partial}{\partial y^j}+X^j\frac{\partial}{\partial y^j}$ is taken, $X=X^i\left(\frac{\partial}{\partial x^i}-N_i^j(x,y)\frac{\partial}{\partial y^j}\right)+\left(N_i^j(x,y)X^i+X^j\right)\frac{\partial}{\partial y^j}=X^i\frac{\delta}{\delta x^i}+\tilde{X}^j\frac{\partial}{\partial y^j}$is obtained. Here, $\frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial y^j}$ are bases of $\Gamma\left(HTM_{|u|}\right)$ and $\Gamma\left(VTM_{|u|}\right)$ respectively. Their dual bases are, meanly the bases of $\left(\Gamma\left(HTM_{|u|}\right)\right)^*$ and $\left(\Gamma\left(VTM_{|u|}\right)\right)^*$ are dx^i and$\delta y^j=dy^j+N_i^jdx^i$, respectively.
For $\eta\in\left(\Gamma\left(TTM_{|u|}\right)\right)^*$, if $\eta=\eta_idx^i+\eta_jdy^j=dx^i+\eta_j\left(N_i^jdx^i+dy^j\right)=\tilde{\eta_i}dx^i+\eta_j\delta y^j=\eta^H+\eta^V$, then $\eta^H\in\left(\Gamma\left(HTM_{|u|}\right)\right)^*$ and $\eta^V\in\left(\Gamma\left(VTM_{|u|}\right)\right)^*$.
Also, following relations hold;
$\langle\frac{\delta}{\delta x^i},dx^j\rangle=\delta_{ij}$, $\langle\frac{\delta}{\delta x^i},\delta y^j\rangle=0$, $\langle\frac{\partial}{\partial y^i},\delta y^j\rangle=\delta_{ij},\langle\frac{\partial}{\partial y^i},dy^j\rangle=0$.
There is a Riemannian metric on TM_0 with almost product structure Q with the properties $Q\left(\frac{\partial}{\partial y^i}\right)=\frac{\delta}{\delta x^i}$ and $Q\left(\frac{\delta}{\delta x^i}\right)=\frac{\partial}{\partial y^i}$[2].

The Riemannian metric G is obtained from a Finsler metric and it is called the Sasaki-Finsler metric on TM_0 and its distributions are as follows:
$G(X,Y) = G(X^H,Y^H) + G(X^V,Y^V) = G^H(X,Y) + G^V(X,Y)$ where $X,Y \in \Gamma(TTM_{|u|})$, $X^H,Y^H \epsilon \Gamma(HTM_{|u|})$ and $X^V,Y^V \epsilon \Gamma(VTM_{|u|})$. Here G^H and G^V are Riemannian metrics of type of $\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$ on HTM and VTM, respectively.
$G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right) = g_{ij}$ and $G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right) = 0$.

$(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are m -dimensional Sasakian structures on HTM and VTM, respectively. Here m-dimentional base space M is a Finsler manifold and G is a Finsler metric on TM.

For all vector fields $X^H,Y^H,\xi^H \epsilon \Gamma(HTM_{|u|})$ and $X^V,Y^V,\xi^V \epsilon \Gamma(VTM_{|u|})$, where; Φ denotes a tensor field of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, ξ is a structure vector field of type $\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$, η is an 1-form of type $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, G is a Sasaki-Finsler metric structure of type $\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}$, ∇ is the Finsler connection with respect to G on TM, L is the Lie differential operator, R is the Riemann curvature tensor field of type $\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix}$ of the manifold, S is the Ricci tensor field of type $\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}$.

The following relations hold for Sasakian structures on HTM and VTM[5]:
$\Phi.\Phi = -I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V$(2.1)
$\Phi\xi^H = 0$, $\Phi\xi^V = 0$(2.2)
$\eta^H(\xi^H) + \eta^V(\xi^V) = 1$ (2.3)$\eta^H(\Phi X^H) = 0, \eta^V(\Phi X^V) = 0$(2.4)
$G^H(\Phi X,\Phi Y) = G^H(X,Y) - \eta^H(X^H)\eta^H(Y^H), G^V(\Phi X,\Phi Y) = G^V(X,Y) - \eta^V(X^V)\eta^V(Y^V)$(2.5)
$G^H(X,\xi) = \eta^H(X^H),\ G^V(X,\xi) = \eta^V(X^V)$ (2.6)
$\nabla^H_\xi \Phi = 0,\ \nabla^V_\xi \Phi = 0$ (2.7)
$\nabla^H_X \xi^H = -\frac{1}{2}\Phi X^H,\ \nabla^V_X \xi^V = -\frac{1}{2}\Phi X^V$ (2.8)
$(\nabla^H_X \Phi)Y^H = \frac{1}{2}[G^H(X^H,Y^H)\xi^H - \eta^H(Y^H)X^H]$,
$(\nabla^V_X \Phi)Y^V = \frac{1}{2}[G^V(X^V,Y^V)\xi^V - \eta^V(Y^V)X^V]$ (2.9)
$R(X^H,Y^H)\xi^H = \frac{1}{4}[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H]$,
$R(X^V,Y^V)\xi^V = \frac{1}{4}[\eta^V(Y^V)X^V - \eta^V(X^V)Y^V]$(2.10)
$R(X^H,\xi^H)Y^H = \frac{1}{4}[\eta^H(Y^H)X^H - G^H(X,Y)\xi^H]$,
$R(X^V,\xi^V)Y^V = \frac{1}{4}[\eta^V(Y^V)X^V - G^V(X,Y)\xi^V]$(2.11)
$S(\xi^H,\xi^H) = \frac{n}{2},\ S(\xi^V,\xi^V) = \frac{n}{2}$(2.12)
$S(X,\xi) = S(X^H,\xi^H) + S(X^V,\xi^V) = \frac{n}{2}(\eta^H(X^H) + \eta^V(X^V))$(2.13)
For a Sasakian Finsler metric manifold HTM, quasi-conformal curvature tensor is described as follows:
$C^*(X^H,Y^H)Z^H = aR(X^H,Y^H)Z^H + b[S(Y^H,Z^H)X^H - S(X^H,Z^H)Y^H + G(Y^H,Z^H)QX^H - G(X^H,Z^H)QY^H] - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)(G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H)$ (2.14)
for vector fields $X^H,Y^H,Z^H \epsilon \Gamma(HTM_{|u|})$, where r is the scalar curvature tensor of TM and a and b are constants.
If $a = 1$ and $= -\frac{1}{2n-1}$, (2.14) takes the following form:
$$C^*(X^H,Y^H)Z^H = R(X^H,Y^H)Z^H + \frac{1}{2n-1}[S(Y^H,Z^H)X^H - S(X^H,Z^H)Y^H + G(Y^H,Z^H)QX^H - G(X^H,Z^H)QY^H] + \frac{r}{(2n)(2n-1)}(G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H) = C(X^H,Y^H)Z^H$$
where C is conformal curvature tensor.
If $C^* = 0$ then Sasakian Finsler metric manifold (HTM,G) is quasi-conformally flat. If
$S(X^H,Y^H) = aG(X^H,Y^H) + b\eta^H(X^H)\eta^H(Y^H)$(2.15)
for $X^H,Y^H \epsilon T_u{}^H TM$ and where a,b are constants, then Sasakian Finsler metric manifold (HTM,G) is an η-Einstein manifold.

III. QUASI-CONFORMALLY FLAT SASAKIAN STRUCTURES $(\phi^H,\xi^H,\eta^H,G^H)$ON HTM

For a quasi-conformally flat Sasakian Finsler manifold from (2.14) we get the following
$G(R(X^H,Y^H)Z^H,W^H) = \frac{b}{a}[S(X^H,Z^H)G(Y^H,W^H) - S(Y^H,Z^H)G(X^H,W^H) + S(Y^H,W^H)G(X^H,W^H) - S(X^H,W^H)G(Y^H,Z^H)] + \frac{r}{a(2n+1)}\left(\frac{a}{2n} + 2b\right)(G(Y^H,Z^H)G(X^H,W^H) - G(X^H,Z^H)G(Y^H,W^H))$ (3.1)
where a and b are constants.
Putting $Z^H = \xi^H$ in (3.1), with the help of (2.3), (2.6) and (2.13) we get the following;
$G(R(X^H,Y^H)\xi^H) - \frac{b}{a}\left[\frac{n}{2}\eta^H(X^H)G(Y^H,W^H) - \frac{n}{2}\eta^H(Y^H)G(X^H,W^H) + S(Y^H,W^H)\eta^H(X^H) - S(X^H,W^H)\eta^H(Y^H)\right] - \frac{r}{a(2n+1)}\left(\frac{a}{2n} + 2b\right)(\eta^H(Y^H)G(X^H,W^H) - \eta^H(X^H)G(Y^H,W^H)) = 0$ (3.2)
Putting $X^H = \xi^H$ in (3.2), by using (2.3), (2.6), (2.10) and (2.13) we have
$S(Y^H,W^H) = \lambda G(Y^H,W^H) + \mu\eta^H(Y^H)\eta^H(W^H)$(3.3)
where
$\lambda = \left[-\frac{n}{2} + \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - \frac{a}{4b}\right]$ (3.4)
and

$\mu = \left[n - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) + \frac{a}{4b}\right]$. (3.5)

Therefore, due to $\lambda + \mu = \frac{n}{2}$, it is possible to write the theorem below:

Theorem 3.1 If $(HTM, \phi^H, \xi^H, \eta^H, G^H)$ is quasi-conformally flat then it is an η-Einstein manifold.

Corollary 3.1 A compact orientable quasi-conformally flat Sasakian Finsler manifold $(HTM, \phi H, \xi H, \eta H, GH$does not admit a nonisometric conformal transformation.

Putting $Y^H = W^H = E_i^H$ in (3.3) and taking summation over $1 \leq i \leq 2n + 1$, so

$r = \lambda(2n + 1) + \mu$. (3.6)

By using (3.4) and (3.5) in (3.6), we have

$\left[\frac{r}{(2n+1)} - \frac{n}{2}\right]\left[\frac{a}{b} + (2n + 1)\right] = 0$. (3.7)

This means

$r = \frac{n}{2}(2n + 1)$ (3.8)

or

$b = \frac{a}{(1-2n)}$ (3.9)

By using (3.9) in (2.14), we get

$C^*(X^H, Y^H)Z^H = aC(X^H, Y^H)Z^H$ (3.10)

where C is conformal curvature tensor. In this case conformally flatness and quasi-conformally flatness are equivalent. A $(2n + 1)$-dimensional conformally flat Sasakian Finsler manifold $(HTM, \phi^H, \xi^H, \eta^H, G^H)$ is of constant curvature and also a manifold of constant curvature is conformally flat. So a Sasakian Finsler manifold$(HTM, \phi^H, \xi^H, \eta^H, G^H)$ is conformally if it is locally isometric to $S^{2n+1}(1)$ and vice versa.

With the help of (3.3), (3.4), (3.5) and (3.8) we obtain

$S(Y^H, W^H) = \frac{n}{2}G(Y^H, W^H)$. (3.11)

Hence Sasakian Finsler manifold HTM is an Einstein manifold. By using (3.8), (3.9) and (3.11) in (3.1), we get

$G(R(X^H, Y^H)Z^H, W^H) = \frac{1}{4}\big(G(Y^H, Z^H)G(X^H, W^H) - GXH, ZHGYH, WH.$

Then Sasakian Finsler manifold HTM is of constant curvature. So it is possible to say it is locally isometric to sphere $S^{2n+1}(1)$. In this case HTM is quasi-conformally flat. These enable us to establish the theorem below:

Theorem 3.2 A $(2n + 1)$ -dimensional $(n > 2)$ Sasakian Finsler manifold is quasi-conformally flat if it is locally isometric to sphere $S^{2n+1}(1)$ and vice versa.

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