# POST-NEWTONIAN LOGARITHMIC CORRECTION TERM EFFECTS IN TWO-BODY DYNAMICS

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**Abstract** - We use the logarithmic correction to the gravitational potential as introduced by Mucket and Treder, in order to study the motion of a secondary celestial body under the influence of the corrected gravitational force of the primary. We derive two equations to approximate the periastron time rate of change and its total variation over one revolution (i.e., the difference between the anomalistic period and the Keplerian period) under the influence of the non-Newtonian radial acceleration. In a kinematic sense, this influence produces apsidal motion. We performed numerical estimations for Mercury and for the companion star of the pulsar PSR 1913+16 and the extra-solar planet b orbiting the star HD 80606 in the constellation of Ursa-Major. We also considered the case of the artificial Earth satellite GRACE-A, but the results present a low degree of reliability from a practical standpoint.

**Keywords** - Logarithmic Potential, Gauss' Planetary Equations, Periastron Time, Anomalistic Period.

## I. INTRODUCTION

In order to explain Mercury's difference between observed and theoretical perihelion rates, different theories were proposed, theories that were related to modified versions of the Newtonian potential. Recently in 1977 Mucket and Treder [1977] introduced a logarithmic correction to the gravitational potential per unit mass that has the following form:

$$V(r)=-\frac{G_{\infty}M_p}{r}(1+\alpha+\alpha\ln r). \quad (1)$$

where $M_p$ is the mass of the primary and r is the radial distance from the center of the primary and α is small numerical parameter such that $|\alpha|<<1$. In this kind of potential, the potential energy V is of real value and also analytic, possessing an isolated singularity at r= 0, and the point r = (0, 0) corresponds to a collision of the secondary with the primary (Ballinger and Diacu, 1992). For this kind of potential, the corresponding force per unit mass becomes:

$$F(r)=-\frac{\partial V(r)}{\partial r}=-\frac{G_{\infty}M_p}{r^2}(1+\alpha\ln r). \quad (2)$$

In this paper, we deal with the logarithmic correction introduced to the Newtonian potential by Mucker and Treder (1977) and its effect on satellites. By evaluating the corresponding logarithmic correction to the potential on the Keplerian and on the precessing Keplerian orbital ellipses respectively, we derive the Lagrange planetary equations up to a first order perturbation. Finally, we apply the derived equations to estimate the corresponding time rates of change of the elements for an orbit reported by GRACE mission

## II. PERTURBATIONS AND LAGRANGE'S PLANETARY EQUATIONS

In a two-body problem the secondary body moves under the dominant force of the primary one. However, other bodies exert forces, which depend on the relative positions of the objects and disturb the motion of the two-body system. The resulting deviations from the unperturbed orbit are usually very small. Given the elements describing the relative position of the two bodies at any instant, the perturbations can be calculated as the time rates of change of these elements. If $R\neq 0$ is the disturbing function, one may use the Lagrange equations to estimate these time rates of change. The Lagrange equations are written as follows (Kaula, 2000; Vallado, 2007).

$$\frac{da}{dt}=\frac{2}{na}\frac{\partial R}{\partial M}, \quad (3)$$

$$\frac{de}{dt}=\frac{\left(1-e^2\right)}{na^2e}\frac{\partial R}{\partial M}-\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega}, \quad (4)$$

$$\frac{d\omega}{dt}=\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e}-\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}, \quad (5)$$

$$\frac{di}{dt}=\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \omega}-\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} \quad (6)$$

$$\frac{d\Omega}{dt}=\frac{1}{n\,a^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}, \quad (7)$$

$$\frac{dM}{dt}=n-\frac{\left(1-e^2\right)}{na^2e}\frac{\partial R}{\partial e}-\frac{2}{na}\frac{\partial R}{\partial a} \quad (8)$$

where $a$ is the semimajor axis of the orbit of the secondary body, $e$ is its eccentricity, $i$ is its orbital inclination, $\Omega$ is the longitude of the node, $\omega$ is the argument of the perigee of the orbiting body,

$n = \sqrt{G_\infty M_p}\, a^{-3/2}$ is the mean motion and $M$ is the mean anomaly. In order to use Eqs (6)–(11) the disturbing function $R$ must be written in terms of the orbital elements.

## III. THE LOGARITHMIC PERTURBING FUNCTION AND THE LAGRANGE EQUATIONS

In the case of an Earth orbiting satellite, the logarithmic perturbing potential per unit of mass can be written as follows (Fischbach and Talmadge, 1999)

$$R_{\log} = -\frac{\mu_p \alpha}{r}\left(1 + \ln r\right) \tag{9}$$

where $\mu_p = GM_p$. For such a satellite orbiting the Earth on a Keplerian ellipse with $a \approx \lambda$, we can find that, on the unperturbed Keplerian ellipse, this disturbing function is expressed as follows:

$$R_{\log} = -\frac{\mu_p \alpha\left[1 + \left(1 + e\cos(\omega + f)\right)\right]}{a\left(1 - e^2\right)}\left[1 + \ln\left(\frac{a\left(1 - e^2\right)}{\left(1 - e\cos(\omega + f)\right)}\right)\right]. \tag{10}$$

Then, we evaluate the partial derivatives of $R_{Yk}$ with respect to $a$, $e$, $\omega$, $i$, $\Omega$ and $M$. In order to calculate $\partial R_{Yk} / \partial M$ we have to express the true anomaly $f$ as a function of $M$. This is accomplished by using the following series expansion (Murray and Dermott, 1999):

$$f \cong M + 2e\sin\left(M\right) + \frac{5}{4}e^2 \sin\left(2M\right) + \mathrm{O}(e^3). \tag{11}$$

Following these evaluations (we cut out the details), we replace the corresponding derivatives $\partial R_{Yk} / \partial s$, $s \in \{a,\ e,\ \omega,\ i,\ \Omega,\ M\}$, for $\partial R / \partial s$ to Eqs (6)–(11) after setting $a = a_0,\ e = e_0,\ \omega = \omega_0,\ i = i_0,\ \Omega = \Omega_0,\ M = M_0$ in order to obtain the Lagrange equations for the time rates of the orbital elements up to a first order perturbation. Finally, we have that these equations are:

$$\frac{da}{dt} = \frac{2\mu_p \alpha e_0 B_0 \sin A_0}{n_0 a_0^2\left(1 - e_0^2\right)} \ln\left(C_0\right) \tag{12}$$

$$\frac{de}{dt} = \frac{\mu_p \alpha B_0}{n_0 a_0^3} \sin A_0 \ln\left(C_0\right)\left[1 + \frac{1}{\sqrt{1 - e_0^2}}\right] \tag{13}$$

$$\frac{d\omega}{dt} = -\frac{\mu_p \alpha}{n_0 a_0^4 e_0\left(1 - e_0^2\right)^{3/2}}\left[e_0\left(1 + e\cos A_0\right)\left(F_0 + e_0\left(\begin{matrix} F_0 \cos A_0 \\ + 2a_0\left(1 + e_0 \ln C_0\right)\end{matrix}\right)\right)\right] - \frac{\mu_p \alpha D_0}{n_0 a_0^3 e_0 \sqrt{1 - e_0^2}}\left(1 + e_0 \ln\left(C_0\right)\right)$$

(14)

$$\frac{di}{dt} = \frac{\cot i_0 \mu_p \alpha e_0}{n_0 a_0^3\left(1 - e_0^2\right)^{3/2}} \sin A_0 \ln\left(C_0\right) \tag{15}$$

$$\frac{d\Omega}{dt} = 0 \tag{16}$$

$$\frac{dM}{dt} = n_0 + \frac{\mu_p \alpha}{n_0 a_0^3 e_0}\left[D_0 \ln C_0 + \frac{a_0}{C_0^2}\left(F_0\left(1 - e_0^2\right) + 2e_0 C_0^3\right)\right] \tag{17}$$

where

$$A_0 = \omega_0 + n_0 t + 2e_0 \sin\left(n_0 t\right) + \frac{5e_0^2}{4}\sin\left(2n_0 t\right) \tag{18}$$

$$B_0 = 1 + 2e_0 \cos\left(n_0 t\right) + \frac{5e_0^2}{2}\cos\left(2n_0 t\right) \tag{19}$$

$$C_0 = \frac{a_0\left(1 - e_0^2\right)}{1 + e_0 \cos A_0} \tag{20}$$

$$D_0 = \left(\left(\cos A_0 - e_0 A_0\left(2\sin\left(n_0 t\right) + \frac{5e_0}{2}\sin\left(2n_0 t\right)\right)\sin A_0\right)\right) \tag{21}$$

$$E_0 = \frac{2e_0\left(1+e_0\cos A_0\right)}{a_0\left(1-e_0^2\right)^2} = \frac{2e_0}{C_0\left(1-e_0^2\right)} \tag{22}$$

$$F_0 = \left(-\frac{2e_0a_0}{1+e_0\cos A_0} - \frac{D_0a_0\left(1-e_0^2\right)}{\left(1+e_0\cos A\right)^2}\right) = -C_0\left(\frac{2e_0}{\left(1-e_0^2\right)} + \frac{D_0}{\left(1+e_0\cos A_0\right)}\right) \tag{23}$$

Via the procedure described above, the disturbing function is approximated by:

$$R_{Sel} = -\frac{\mu_p}{C_0}\left(1+\ln C_0\right) \tag{24}$$

## V. THE PRECESSING PERTURBING FUNCTION AND THE LAGRANGE EQUATIONS

Next, we derive the Lagrange equations on the precessing orbital ellipse. In this case, the motion of the secondary body can be described by the following Binet-type differential equation (Mucker, and Treder, 1977)

$$\ddot{u}+\left(1+\alpha\right)u = \frac{G_\infty M_p\left(1+\alpha\right)}{h^2} - \frac{\alpha G_\infty M_p}{h^2}\ln\left(\frac{G_\infty M_p}{h^2}\right) \tag{25}$$

where $u = 1/r$, $\theta = \omega + f$ and h is the angular momentum of the body. The first order solution of this equation is the following (ibid, 1977):

$$r = \frac{a\left(1-e_0^2\right)}{\left(1+\alpha+\alpha\ln a\right)\left[1+e\cos\left[q\left(\omega+f\right)\right]\right]} \tag{26}$$

where $a_0$ is the semimajor axis of the orbit and (ibid, 1977):

$$q = \left(1+\frac{\alpha}{2}\right) \tag{27}$$

Eq. (27) represents a precessing ellipse radial vector $q_0 \neq 1$. Then, the Yukawa disturbing potential on this ellipse in the following way is

$$R_{\log} = -\frac{\mu_P\alpha Q\left(\alpha\right)\left[1+\left[1+e\cos\left(q\left(\omega+f\right)\right)\right]\right]}{a\left(1-e^2\right)}\left[1+\ln\left(\frac{a\left(1-e^2\right)}{Q\left(\alpha\right)\left(1-e\cos\left(q\left(\omega+f\right)\right)\right)}\right)\right] \tag{28}$$

where

$$Q\left(\alpha\right) = \left(1+\alpha\left(1+\ln a\right)\right)^{-1}. \tag{29}$$

If q = 1 and α = 0 Eq. (27) gives back the unperturbed Keplerian equation for the radius vector of the secondary body-

By following a procedure like the one described in the previous section, we obtain that, on the precessing Keplerian ellipse, the time rates of the orbital elements are:

$$\frac{da}{dt} = \frac{2\mu_p e_0\alpha q_0}{n_0a_0^2\left(1-e_0^2\right)Q_0}B_0\sin\left(q_0A_0\right)\ln\left(Q_0C_0\right) \tag{30}$$

$$\frac{de}{dt} = \frac{\mu_p q_0\alpha e_0}{n_0e_0a_0^2\left(1-e_0^2\right)Q_0}\sin A_0\left(-\sqrt{1-e_0^2}+\left(1-e_0^2\right)B_0\right)\ln\left(Q_0C_0\right) \tag{31}$$

$$\frac{d\omega}{dt} = -\frac{\mu_p\alpha\sqrt{1-e_0^2}}{n_0e_0a_0^2Q_0}\left[e^2X_0+\left(e_0+N_0\right)\left(1+\ln\left(Q_0C_0\right)\right)\right] \tag{32}$$

$$\frac{di}{dt} = \frac{\mu_p e_0\alpha q_0\cot i_0}{n_0a_0^3Q_0\left(1-e_0^2\right)^{3/2}}\sin\left(q_0A_0\right)\ln\left(C_0Q_0\right) \tag{33}$$

$$\frac{dM}{dt} = n_0 + \frac{2\mu_p\alpha}{n_0 a_0 Q_0}\left[\frac{\alpha}{QC_0} + \frac{\left(1+\ln\left(C_0Q_0\right)\right)}{C_0} - \frac{e^2C_0\left(Q_0+\alpha\right)}{Q_0 a_0}\right] - \frac{\mu_p\alpha\left(1-e_0^2\right)}{n_0 e_0 a_0^2 Q_0}\left[e_0^2 X_0 + e_0\left(1+\ln\left(Q_0C_0\right)\right)\right] \quad (34)$$

while the disturbing function R is approximated by

$$R_{PYk} = -\frac{\mu_p\alpha}{\lambda B_1}e^{-B_1} \quad (35)$$

where

$$A_1 = 1+\alpha e^{-\frac{a_0}{\lambda}}\left(1+\frac{a_0}{\lambda}-\frac{a_0^2}{\lambda^2}\right) \quad (36)$$

$$B_1 = \frac{a_0\left(1-e_0^2\right)}{\lambda A_1\left(1+e_0\cos\left(q_0F_0\right)\right)} \quad (37)$$

## VII. NUMERICAL RESULTS

Next, we apply Eqs (15)-(20) and (29)-(34) to calculate the time rates of the orbital elements for the satellite GRACE-A. GRACE-A is an Earth orbiting satellite that is a part of the Gravity Recovery and Climate Experiment. The data that are reported in the website of this experiment (http://www.csr.utexas.edu/grace) are: $a_0 = 6876.4816,\ e_0 = 0.00040989,$ and therefore $n_0 = 0.001100118$ rad/s $= 15.113$rev/d, $i_0 = 89.025446^{\circ},\quad \omega_0 = 302.414244^{\circ},\quad \Omega_0 = 354.447149^{\circ},\quad M_0 = 80.713591^{\circ}.$ Unfortunately, no improved estimates for $\lambda$ and $\alpha$ from GRACE mission are not available yet (Iorio, 2002). Following Lucchesi (2003), we have used for $\lambda$ the maximum distance range $1.073R_E = 6843.737$ km while, for the coupling constant $\alpha$, the maximum value $4.6\times10^{-10}$ and the value $4.\times10^{-12}$ have been utilized. Our results are given in Tables 1 and 2 below.

| GRACE-A | | | | | | |
|---|---|---|---|---|---|---|
| $\alpha = 10^{-9}$ | $\frac{da}{dt}$ [m/d] | $\frac{de}{dt}$ [d$^{-1}$] | $\frac{d\omega}{dt}$ [°/d] | $\frac{di}{dt}$ [°/d] | $\frac{d\Omega}{dt}$ [°/d] | $\frac{dM}{dt}$ [rev/d] |
| GRACE-A | $-7.122\times10^{-3}$ | $-1.035\times10^{-9}$ | $-3.425\times10^{-5}$ | $-1.535\times10^{-13}$ | 0 | $2.919\times10^{-4}$ |
| Mercury | 66.24 | $8.703\times10^{-10}$ | $-1.044\times10^{-10}$ | $5.475\times10^{-11}$ | 0 | $2.921\times10^{-10}$ |
| Extra-Solar Planet | 0.0184028 | $8.323\times10^{-15}$ | $-1.800\times10^{-9}$ | $1.630\times10^{-17}$ | 0 | $1.923\times10^{-10}$ |
| $\alpha = 10^{-10}$ | | | | | | |
| GRACE-A | $-7.122\times10^{-4}$ | $-1.035\times10^{-10}$ | $-3.425\times10^{-6}$ | $-1.535\times10^{-14}$ | 0 | $2.919\times10^{-5}$ |
| Mercury | 6.24 | $8.703\times10^{-11}$ | $-1.044\times10^{-11}$ | $5.475\times10^{-12}$ | 0 | $2.921\times10^{-11}$ |
| Extra-Solar Planet | 0.00184028 | $8.323\times10^{-16}$ | $-1.800\times10^{-10}$ | $1.630\times10^{-17}$ | 0 | $1.923\times10^{-11}$ |

**Table 1. First order perturbation effects of the Yukawa correction on the time rates of the orbital elements of the satellite GRACE-A calculated on the unperturbed Keplerian ellipse.**

| GRACE-A | | | | | | |
|---|---|---|---|---|---|---|
| $\lambda \cong 1.073R_E$ | $\frac{da}{dt}$ [m/d] | $\frac{de}{dt}$ [d$^{-1}$] | $\frac{d\omega}{dt}$ [°/d] | $\frac{di}{dt}$ [°/d] | $\frac{d\Omega}{dt}$ [°/d] | $\frac{dM}{dt}$ [rev/d] |
| $\alpha = 4.6\times10^{-10}$ | $-1.530\times10^{-4}$ | $-2.220\times10^{-11}$ | $-0.0024140200$ | $-1.152\times10^{-11}$ | 0 | $6.700\times10^{-6}$ |
| $\alpha = 4.0\times10^{-12}$ | $-1.330\times10^{-6}$ | $-1.931\times10^{-13}$ | $-0.0000209915$ | $-9.406\times10^{-14}$ | 0 | $5.830\times10^{-8}$ |

**Table 2. First order perturbation effects of the Yukawa correction on the time rates of the orbital elements of the satellite GRACE-A calculated on the precessing Keplerian ellipse.**

Comparing the corresponding entries in Tables 1 and 2, we can see that there is not significant difference between the orbital time rates of change on the Keplerian ellipse and those on the precessing one. The orbital element most affected is the argument of the perigee. For the same value of $\alpha$, there is, approximately, a 0.05% difference between the respective rates of change on the Keplerian and the

precessing Keplerian ellipses. It is also seen that, in both cases, the higher value of $\alpha$ results higher rates of change than the lower one does. Again, among all the orbital elements, the argument of the perigee is affected the most. Similarly, for the same values of $\alpha$ the mean anomaly time rates of change calculated on the Keplerian ellipse and the precessing one are almost identical. In a year's time, the total effect of the Yukawa correction on the perigee will be of about $-0.880^{o}$ on the Keplerian ellipse and $-0.881^{o}$ on the precessing one, respectively, when $\alpha = 4.6\times10^{-10}$. This effect could be possibly be detected by today's technology. In the nearest future, when new values for $\alpha$ and $\lambda$ will be obtained from GRACE mission itself, these effects can be more accurately predicted.

## VIII. CONCLUSION

For a Yukawa type correction to the gravitational potential, which is predicted by today's various gravitational theories, the Lagrange planetary equations are used in order to study the motion of an orbiting body in this type of potential. We have studied the motion of the Earth orbiting satellite GRACE-A, by deriving the rates of change of its orbital elements on the Keplerian ellipse and on the precessing one. We find that, in this case, among all the orbital elements, the argument of the perigee is affected the most. There is no significant difference between these rates on the unperturbed Keplerian ellipse and the precessing one, respectively, except for those related to the argument of the perigee which differ from each other by about 0.05%. For $\lambda \cong 1.073R_E$ a larger value for $b$ produces higher rates of change.

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