POST-NEWTONIAN ANOMALISTIC PERIOD EFFECTS OF CELESTIAL BODIES DUE TO A GRAVITATIONAL LAW WITH A MUCKET-TREDER CORRECTION TERM

[1]IOANNIS HARANAS, [2]IOANNIS GKIGKITZIS, [3]JUSTIN MOFFATT, [4]JASLEEN GOSSAL

[1,3,4]Dept. of Physics and Astronomy, York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada
[2]Departments of Mathematics and Biomedical Physics, East Carolina University 124 Austin Building, East Fifth Street Greenville, NC 27858-4353, USA

Abstract - We use a logarithmic correction to the gravitational potential as introduced by Mücket and Treder (1977), in order to study the motion of a secondary celestial body under the influence of the corrected gravitational force of a primary. We derive two equations to approximate the periastron time rate of change and its total variation over one revolution (i.e., the difference between the anomalistic period and the Keplerian period) under the influence of the non-Newtonian radial acceleration. In a kinematic sense, this influence produces apsidal motion. We performed numerical estimations for Mercury, for the companion star of the pulsar PSR 1913+16 and, also, the extra-solar planet b orbiting the star HD 80606 in the constellation of Ursa Major. We also considered the case of the artificial Earth satellite GRACE-A, but the results present a low degree of reliability from a practical standpoint.

Keywords - Logarithmic Potential, Gauss' Planetary Equations, Periastron Time, Anomalistic Period, Keplerian Period.

I. INTRODUCTION

In order to explain the difference between the Mercury's observed and theoretical perihelion rate of precession, several theories were proposed. These theories are related to modified versions of the Newtonian potential. Mücket and Treder (1977) introduced a logarithmic correction to the gravitational potential per unit mass. Expressing the general problem of the motion of a secondary body in the vicinity of a primary one in terms of the classical two-body model, the corresponding corrected potential energy has the following form:

$$V(r) = -\frac{GM_p}{r}\left(1+\alpha+\alpha\ln r\right) \quad . (1)$$

where M_p is the mass of the primary, G is the Newtonian gravitational constant, r is the radial distance of the secondary body from the primary one and α is a small parameter such that $|\alpha| << 1$. The corresponding force per unit mass becomes:

$$\mathbf{F}(r) = -\frac{\partial V(r)}{\partial r}\mathbf{r} = -\frac{GM_p}{r^2}\left(1+\alpha\ln r\right)\mathbf{r} \, . \quad (2)$$

Various authors have considered the same potential. In Mioc and Blaga (1991) the authors examine the orbital motion of a body with the Mücket-Treder post–Newtonian gravitational law, by working out the changes of the orbital elements over a nodal period. Next, in Diacu (1993) the author proves that for a Mücket – Treder gravitational law, triple collision-ejection orbits also tend to a Newtonian central configuration. Similarly, in Mioc (2004) the author has worked out the symmetries of the Mücket – Treder potential in a two-body scenario.

II. RATE OF CHANGE AND VARIATION PER REVOLUTION OF THE PERIASTRON TIME

Consider the unperturbed relative orbit of the secondary, obviously a Keplerian ellipse. Let a be the semimajor axis, e the eccentricity, n the mean motion, and M the mean anomaly of this orbit. First, we will express the rate of change of the periastron time T_0 in terms of the true anomaly f. The well-known relation

$$M = n\left(t - T_0\right) \quad (3)$$

connects T_0 to M and n. Here t denotes the time variable. We differentiate Eq. (3) with respect to t to obtain (Haranas, Ragos, and Mioc, 2010):

$$\frac{dT_0}{dt} = \frac{\left(t-T_0\right)}{n}\frac{dn}{dt} - \frac{1}{n}\frac{dM}{dt} + 1. \quad (4)$$

Using also that, on the unperturbed Keplerian orbit of the secondary Kepler's third law is expressed by

$$\frac{dn}{dt} = -\frac{3n}{2a}\frac{da}{dt} \quad (5)$$

In the presence of a perturbation, the rates of change of the orbital elements can be found by means of Gauss' planetary equations: For the semimajor axis and the mean anomaly they read:

$$\frac{da}{dt}=\frac{2}{n\sqrt{1-e^2}}\left[e\sin f\,R+\frac{a\left(1-e^2\right)}{r}T\right], \tag{6}$$

$$\frac{dM}{dt}=n-\frac{\left(1-e^2\right)}{nae}\left[\left(\frac{2er}{a\left(1-e^2\right)}-\cos f\right)R+\sin f\left(1+\frac{r}{a\left(1-e^2\right)}\right)T\right]. \tag{7}$$

where R and T are the radial and transverse components of the perturbing acceleration. In our case, $T_{\ln}=0$ while

$$R_{\ln}=-\frac{GM\alpha}{r^2}\ln r\,. \tag{8}$$

Substititing Eqs. (5) – (7) into (4) we obtain that:

$$\frac{dT_0}{dt}=\left[\frac{2r}{n^2a^2}-\frac{\left(1-e^2\right)\cos f}{n^2ea}-\frac{3e\left(t-T_0\right)\sin f}{2na\sqrt{1-e^2}}\right]R_{\ln} \tag{9}$$

where, r, f are connected through the equation:

$$r=\frac{a\left(1-e^2\right)}{1+e\cos f}\,. \tag{10}$$

Then, we use the well-known relations (see, e.g., Murray and Dermott 1999)

$$r=a\left(1-e\cos E\right), \tag{11}$$

$$\frac{dE}{dt}=\frac{n}{1-e\cos E}, \tag{12}$$

$$t-T_0=\frac{E-e\sin E}{n}, \tag{13}$$

$$\cos f=\frac{\cos E-e}{1-e\cos E}=\frac{a\left(\cos E-e\right)}{r}, \tag{14}$$

$$\sin f=\frac{\sqrt{1-e^2}\sin E}{1-e\cos E}=\frac{a\sqrt{1-e^2}\sin E}{r}. \tag{15}$$

to express Eqs. (8)-(9) in terms of the eccentric anomaly E We obtain that:

$$R_{\ln}=-\frac{GM_p\alpha}{a^2\left(1-e\cos E\right)^2}\ln\left[a\left(1-e\cos E\right)\right]=-\frac{\alpha n^2a}{\left(1-e\cos E\right)^2}\ln\left[a\left(1-e\cos E\right)\right] \tag{16}$$

and therefore the final equation for the change of the passage time with respect to the eccentric anomaly becomes:

$$\frac{dT_0}{dE}=-\frac{\alpha\ln\left[a\left(1-e\cos E\right)\right]}{n}\left[2-\frac{\left(1-e^2\right)\left(\cos E-e\right)}{e\left(1-e\cos E\right)^2}-\frac{3e\sin E\left(E-e\sin E\right)}{\left(1-e\cos E\right)^2}\right] \tag{17}$$

The logarithmic correction effect on T_0 over a whole revolution of the secondary is obtained by integrating dT_0/dE over the interval $[0,2\pi]$. To accomplish this we have to approximate the terms:

$\ln\left[a\left(1-e\cos E\right)\right]$, $\left(1-e\cos E\right)^{-2}$. We use the following Taylor expansions of up to second order terms of e:

$$\ln\left[a\left(1-e\cos E\right)\right]\cong \ln a-e\cos E-\frac{1}{2}e^2\cos^2 E+O(e^3),, \quad (18)$$

$$\left(1-e\cos E\right)^{-2}\cong 1+2e\cos E+3e^2\cos^2 E+O\left(e^3\right),. \quad (19)$$

Substituting the approximations (18)-(19) in Eq. (17) we obtain the following equation:

$$\Delta T_0\cong -\int_0^{2\pi}\frac{\alpha}{n}\left[\left(\ln a-e\cos E-\frac{1}{2}e^2\cos^2 E\right)\left(\begin{array}{l}2-\dfrac{\left(1-e^2\right)\left(\cos E-e\right)\left(1+2e\cos E+3e^2\cos^2 E\right)}{e}\\-3e\sin E\left(E-e\sin E\right)\left(1+2e\cos E+3e^2\cos^2 E\right)\end{array}\right)\right]dE. \quad (20)$$

Next, integrating over one cycle and retaining up to six order terms in e, we obtain the following relation:

$$\Delta T_0\cong -\frac{\alpha}{n}\left[\frac{1}{2}+2\ln a+3e\ln a-\frac{3e^2}{2}\left(1-3\ln a\right)-\frac{5e^3}{2}\left(1-\frac{6}{5}\ln a\right)-\frac{3e^4}{8}\left(\frac{232}{3}+\ln a\right)-\frac{9e^5}{10}+\frac{9e^6}{10}\right]. \quad (21)$$

II. NUMERICAL RESULTS

First, we proceed with the calculation of the variation of the perihelion time of Mercury. For this planet, we have used the following orbital parameters: $a=57909083$ km, $e=0.205$, $n=8.07\times10^{-7}$ rad/s. For the primary $M_p=M_{\odot}=1.99\times10^{30}$ kg. For α's value, we have utilized an α in the range $-10^{-8}<\alpha<0\ 10^{-9}$ (Diacu, 1993). Therefore using α = -10^{-9} and -10^{-10} we have obtained:

$$\Delta T_{0Mer}=0.701263 \text{ s/rev} \quad (22)$$

$$\Delta T_{0Mer}=0.0701263 \text{ s/rev} \quad (23)$$

Next, we estimate the variation of the periastron time of the companion star of the pulsar PSR 1913+16. The orbital parameters for this star are $a=1.9501\times10^6$ km, $e=0.617$, and $n=1.575\times10^{-4}$ rad/s. The primary's mass is $M_p=1.387M_{\odot}$. Then, for the values of α mentioned above:

$$\Delta T_{0PSR}=0.00354081 \text{ s/rev} \quad (24)$$

$$\Delta T_{0PSR}=0.000354081 \text{ s/rev} \quad (25)$$

Next, we estimate the variation of the periastron time of the extra-solar planet b companion planet of the star HD 80606 in the constellation of Ursa Major. Planet b is a superjovian planet (eccentric Jupiter) with the following orbital parameters (http://en.wikipedia.org/wiki/HD_80606_b): $a=6.795\times10^7$ km, $e=0.9336$ and $n=6.17\times10^{-7}$ rad/s. Also, $M_{HD}=0.9M_{Sun}=1.791\times10^{30}$ kg. Then we have that:

$$\Delta T_{0b}=2.70172 \text{ s/rev} \quad (27)$$

$$\Delta T_{0b}=0.270172 \text{ s/rev} \quad (28)$$

Finally we have also taken into account the variation of the perigee time of the artificial Earth satellite GRACE-A, with $a=6876.4816$ km, $e=0.00040989$ and $n=0.001100118$ rad/s (http://www.csr.utexas.edu/grace/). We find that:

$$\Delta T_{0b}=1.82802\times10^{-4} \text{ s/rev;} \quad (29)$$

$$\Delta T_{0b}=1.82802\times10^{-5} \text{ s/rev.} \quad (30)$$

But we must say that the results for GRACE-A have a low degree of reliability from a practical standpoint, because of the very small eccentricity. It is known that for quasicircular orbits the position of the periastron (hence the periastron time) cannot be accurately determined. However our determinations are of some interest as regards the magnitude order of the perigee time variation. If we can measure the change in the anomalistic period per revolution for a given body in an elliptical orbit, we can then write that the coupling constant is given by the following expression:

$$\alpha_{el}=-\frac{n_{el}}{\left(400+480e+480e^2+80e^3-640e^4-144e^5+45e^6\right)}\Delta T_{0_{el}}, \quad (31)$$

which for circular orbits simplifies to:

$$\alpha_{cir} = -\frac{n_{cir}}{400}\Delta T_{0_{cir}}, \qquad + \quad (32)$$

and therefore we can further write that:

$$\alpha_{el} = \frac{400}{f(e)}\left(\frac{n_{el}}{n_{cir}}\right)\left(\frac{\Delta T_{0_{el}}}{\Delta T_{0_{cir}}}\right)\alpha_{cir}\,. \quad (33)$$

Where $f(e)$ is the polynomial in eccentricity e of Eq. (31). In figure 1, for $\alpha = -10^{-9}$ we plot the anomalistic period of Mercury as a function of its mean motion n and semimajor axis a, and we see that large values of the semimajor axis and therefore small mean motions correspond to larger advancement of the anomalistic times, the advancement of the anomalistic time reducing as the semimajor axis gets smaller and therefore n increases. Similarly, in figure 2 and for the same α we plot the anomalistic period of companion of the HD 80606 star i.e. planet b as a function of its eccentricity e and semimajor axis a, and we see that the companion time anomalistic period increases drastically as the eccentricity and the semimajor axis increase.

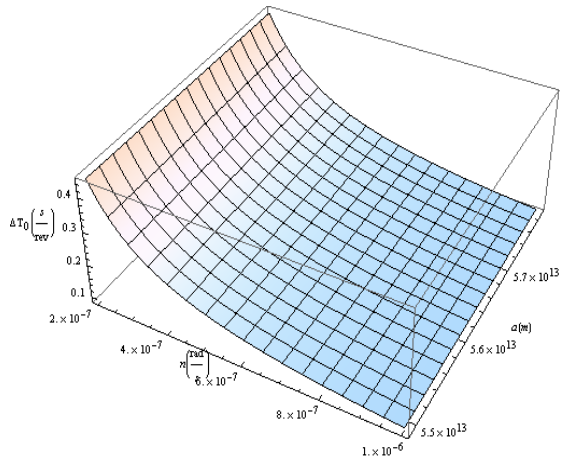


Fig. 1 Plot of the anomalistic period of Mercury versus its semimajor axis a and its mean motion n, for $\alpha = -10^{-9}$.

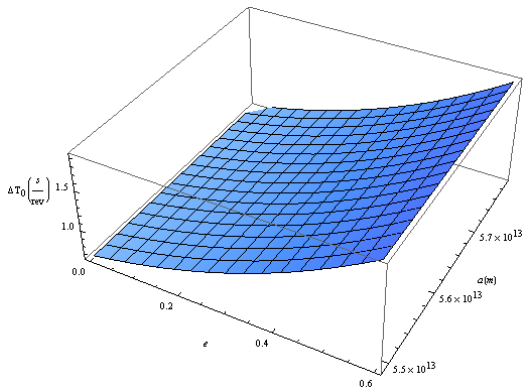


Fig. 2 Plot of the anomalistic period of planet b in the star HD 80606 versus its semimajor axis a and its eccentricity e for $\alpha = -10^{-9}$.

IV. SUMMARY AND CONCLUDING REMARKS

Using a logarithmic correction to the Newtonian gravitational potential as in Mücker and Treder (1977), we derive an eccentric anomaly-dependent equation that estimates the periastron time rate of change in the relative motion of a secondary body orbiting a primary. By using the integral of this equation over a whole revolution, we have found that this correction to the Newtonian potential contributes to an advance of the periastron time, if a negative coupling constant α is used, where a positive coupling constant α results in the recess of the periastron/perihelion time. This variation was estimated for some concrete astronomical situations. A variation for T_0 of the order of few seconds to milliseconds may be detectable by today's technology. This detection can constitute a possible test for the action of Post-Newtonian type forces on the solar system bodies, or on other celestial objects. By no means this is not the only correction to be considered in the modification of the Newtonian gravitational potential. For example general relativistic corrections, as well as quantum corrections can be also considered but that is another topic that we are going to deal with in the nearest future.

REFERENCES

[1] Ballinger, G., and Diacu, F., N., Collision and near collision orbits in post-Newtonian gravitational systems, Roman. Astron. Journal, vol. 3, no.1, 1993.

[2] Diacu, F., N., On the Mücket – Treder gravitational law, preprint DM-621-IR, University of Victoria, Nov. 1992; in "Proceedings of the Conference on Hamiltonian Systems and Celestial Mechanics, Cocoyoc, Mexico, September 1994", (1996) in press.

[3] Haranas, I., Ragos, O. and Mioc V., Yukawa-type Potential Effects in the Anomalistic Period of Celestial Bodies, Astrophys. Space Science 332, 107 (2011).

[4] Mioc, V. and Blaga, P., Orbital motion with the Mücket-Treder post–Newtonian gravitational law, Rom. Astron. Journal, vol. 1, no. 1-2, (1991)

[5] Mioc, V., Symmetries of Mücket – Treder's two-body problem, Hvar Observatory Bulletin, vol. 28, no. 1, pp. 167-179, (2004).

[6] Mücket – Treder J. P., Treder, H. J., The Perihelion Advance According to a Post-Newtonian Gravitational Law with Logarithmic Correction Term, Astron. Nachr. 298, 65 (1977).

[7] Murray, C.D., Dermott, S.F.: Solar System Dynamics. Cambridge University Press (1999).