MESHING OF THREE DIMENSIONAL OBJECTS WITH EFFECTIVE COST OR ENERGY FUNCTION

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Abstract - In this note the process of meshing different types of three-dimensional objects like pentahedron, hexahedron etc. by triangulation with effective energy or cost function will be discussed so that they can be utilized in Engineering and Applied sciences and in day to day life. The methods of meshing will be considered by P-well centered pyramidisation of simple domains in$\mathbb{R}^3$. We discuss the characterization of energy function of three dimensional objects using Base Diagonal Apex Method (BDAMe) and we have shown that the energy function or cost function lies between 0 and 1. We have also developed a modified cost function or energy function for 3-D shapes, for a good quality meshing we give a beautiful theorem.

Keywords - Meshing, N-Simplex, P-Well-Centered, Completely Well-Centered, Space Diagonal, Dihedral, Face Angles, Pentahedron, Hexahedron, Cost Function .

I. INTRODUCTION

In this paper we have demonstrated completely P-well-centered pyramidisation of some simple domains in$\mathbb{R}^3$. There are several types of hexahedron like 7 types of cuboid, pentagonal pyramid, triangular bipyramid etc. Here we have discussed about the meshing of some of those hexahedra into pentahedra which has one quadrilateral face and four triangular faces. Remember that all the hexahedra, discussed here are convex. Three-dimensional simplicial mesh generation aims at tiling a bounded 3D domain with tetrahedrons so that any two of them are either disjoint or sharing a lower dimensional face. The well centeredness of a mesh follows by minimizing a cost function defined over the mesh. Sometimes cost function is referred as the energy function. So all the 3D domains as in here mainly hexahedra are tried to mesh into tetrahedra to minimize the cost function. The meshing of pentahedra are briefly discussed in [1]. Here we discuss the meshing of hexahedra into pentahedra and hence tetrahedra. The characterization of energy function of a pyramid using Base Diagonal Apex Method(BDAMe) are also discussed in [2].

We discuss that in here and also we give another idea to find cost function of pentahedra. In fact the method, we discussed here about the cost function is also applicable for all $(N+1)$-faces polyhedron with N-gonal base provided the circumcenter of the polyhedron lies inside of the polyhedron. In the first section of this paper we have discussed about the meshing of pyramid (pentahedron) with P-well-centered tetrahedra. In the second section we have discussed about the meshing of several types of hexahedra (like Cube, Rectangular cuboid, Pentagonal pyramid) with P-well-centered pentahedra. And in the last section we discuss about characterization of cost function and define an another cost function. Some useful definitions are given in the preliminary section.

II. PRELIMINARY

Before we discuss, we would like to define some basic but important terms.

- A Dihedral angle is the angle of intersection between two planes.
- A Space diagonal of a polyhedron is the line joining two vertices that are not on the same face.
- A Simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimension. Specifically, an n-simplex is an n-dimensional polytope with $(n+1)$vertices, of which the simplex is convex hull. For example, 2-simplex is a tringle, 3-simplex is a tetrahedron, 4-simplex is a pentachoron etc.
- A Completely Well-Centered Mesh is a simplicial mesh in which each simplex contains its circum center in its interior.
- A Completely P-Well-Centered Polyhedron is one for which the circum center of each faces lies inside of those faces. Consider an n-dimensional simplex σ^n embedded in Euclidean space $\mathbb{R}^m$, $m \geq n$. The affine hull of σ^n, $aff(\sigma^n)$, is the smallest affine subspace of $\mathbb{R}^m$ that contains σ^n. In this case, $aff(\sigma^n)$, is a copy of $\mathbb{R}^m$ embedded in $\mathbb{R}^m$. The circum center of σ^n, which we denote $C(\sigma^n)$, is the unique point in $aff(\sigma^n)$ that is equidistant from every vertex of σ^n. For an n-simplex σ^n with $n \geq 3$, it is possible for σ^n to contain its circumcenter $C(\sigma^n)$ while some proper face $\sigma^p \prec \sigma^n$ does not contain its circumcenter $C(\sigma^p)$. It is also possible that for all $1 \leq p < n$ and all $\sigma^p \prec \sigma^n$, $C(\sigma^p)$ lies in the interior of σ^p, but σ^n does not contain its circumcenter. Thus we say that an n-simplexσ^n is a $(p_1, p_2, \dots, p_k)$-well-centered simplex if for $p_i, i = 1,2, \dots k,$ all faces of σ^n of dimension $p_i \leq n$properly contain their circum centers. The

parentheses are suppressed when referring to only one dimension. A simplex σ^n is completely well-centered if it is $(1,2,\dots,n)$-well-centered.

- A Completely Well-Centered Polyhedron is one for which the circum center of each faces lies inside of those faces and circum center of the circumscribed sphere of the polyhedron also lies inside of the polyhedron provided thus sphere exists.

Now we will discuss about the meshing and define a cost function for the shapes.

$1:Meshing of Pyramid (Pentahedron) with P-well-centered Tetrahedrons

Here in this section we are trying to break the pyramids i.e. pentahedra into some completely P-well-centered tetrahedra. We have given two ideas to do so.

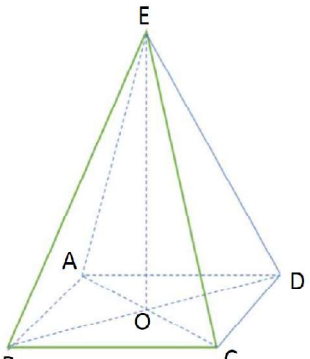


Figure F1

$1.1. We can mesh the pyramid using Base-Diagonal Apex Method(BDAMe). Notice that the base of the pyramid is quadrilateral. In this method we add the two diagonals of the base and add the intersection point of the diagonals to the apex of the pyramid. As shown in the figure F1, using BDAMe, the pyramid ABCDE is subdivided into four tetrahedra namely ABOE, ADOE, DCOE and BCOE, where O is the intersection of the diagonals.

$1.2. In this method we consider such pyramids whose circum center lies inside of the pyramids. Let G be the circum center of the pyramid ABCDE.

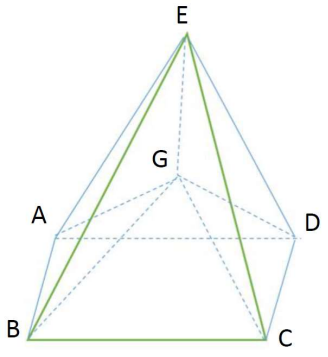


Figure F2

We join AG, BG, CG, DG, EG. It is not necessary that Circum center lies on the base. In this method, as shown in F2, we get four tetrahedra namely ABGE, BCGE, CDGE, DAGE and one copy of pyramid ABCDG with the base ABCD and apex G. Again we can mesh the pyramid ABCDG. We think sometimes this type of meshing is also useful to get smaller pieces of tetrahedra.

$2: Meshing Several Types of Hexahedra with P-Well-Centered Pentahedra and Hence Tetrahedrons

Here we try to mesh several types of hexahedra into P-well-centered pentahedra and as we have already seen the meshing of pentahedra in $1, hence into tetrahedra.

$2.1. Meshing the Cube with P-well-centered Pentahedra: In this sub-section we try to break the cube into some completely P-well-centered pentahedral .

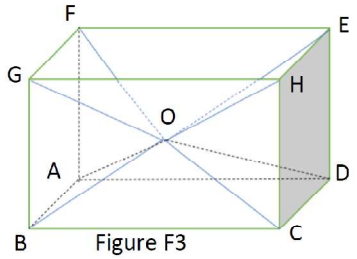


Figure F3

Consider the cube ABCDEFGH and join four space diagonals AH, BE, CF and DG. Their point of intersection is O say. Thus we get six completely P-well-centered pentahedra. The pyramids are ABGFO, ABCDO, BCHGO, CDEHO, ADEFO and EFGHO, as shown in F3. Therefore, the lower bound for this problem is simple to present. The six pyramids are similar to each other that means the face angles and dihedral angles are same for each pyramid.

To calculate the analytical upper bound of the meshing of the cube, we go through some process.

(i) We join the midpoints of the opposite edges for any faces of the cube to get 4 squares and for all six faces we get 24 squares. We refer this process as “splitting the faces”.

(ii) We cut the cube through the midpoints of the edges along coordinate axes to get 8 cubes. We refer this process as “splitting into smaller ones”.

After meshing the cube, we get six P-well-centered square-based pyramids. Now if we perform “splitting the faces” for each face we get total 24 numbers of P-well-centered square-based pyramids. Each pyramid has at least two dihedral angles as 90°. If we perform this process again we get 24×4=96 square-based pyramids but not all are completely P-well-centered. Now after splitting the cube into smaller cubes i.e. after performing “splitting into smaller ones” to get 8 cubes and since each cube is covered by six completely P-well-centered square-based pyramids, we get 48 completely P-well-centered square-based pyramids. Now if we perform “splitting the faces” we get 48×4=192 P-well-centered square-based

pyramids. If we perform 'splitting into smaller ones" twice to get 64cubes then after all this process, we get 64×6×4 =1536 completely P-well-centered square-based pyramids. Similarly, we can mesh the Rectangular Cuboid following those two ways to get rectangle-based completely P-well-centered pentahedral.

$2.2. Meshing the Pentagonal Pyramid: In this subsection we try to mesh the pentagonal pyramid into tetrahedra. We do this in two ways.

(i) We consider a pentagonal pyramid whose circumscribed circle of the base i.e. pentagon exists. For this we can take the base of the pyramid as regular simple pentagon. So taking ABCDEF as pentagonal pyramid, we consider O as the circumcenter of the pentagon ABCDE. Now we connect each vertex of the pentagon with O to get five completely P-well-centered tetrahedra, namely ABOF, BCOF, CDOF, DEOF and EAOF, as shown in F4.

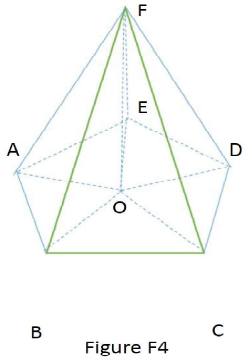


Figure F4

(ii) Now in another way, we consider a pentagonal pyramid whose circum center lies inside of the pyramid. For example, we can take regular pentagonal pyramid or completely well-centered pyramid. Consider G as the center of the circumscribed sphere of the pyramid ABCDEF. We connect each vertex of the pyramid to get five tetrahedra and one pentagonal pyramid. As shown in the figure F5, we get five tetrahedra ABGF, BCGF, CDGF, DEGF, EAGF and one pyramid namely ABCDEG. We can mesh this new pyramid again in these two ways.

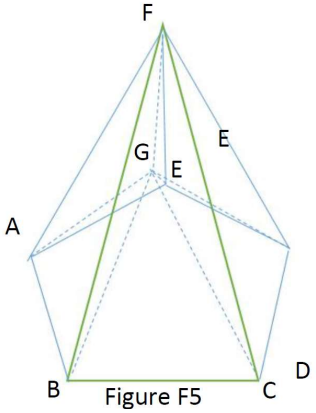


Figure F5

We can extend the second method to mesh a $(N+1)$-sided pyramid with the base N-gon provided that the circumscribed sphere of the pyramidexists and circumcenter lies inside of the pyramid. In the next section we try to define the cost function of N-gonal pyramid if we mesh the pyramid in the second way.

$3: Characterization of Cost Function of a Pyramid

For 3-simplex the cost function or energy function is discussed in [3]. We cannot estimate the energy function for all 3D shapes unless it is meshed into 3-simplex. The energy function of a pentahedron is briefly discussed in [2]. Using BDAMe, if we mesh the pyramid(pentahedron), we get four 3-simplex, which is already discussed in $1. Now each tetrahedron has a cost function. Now to define cost function of 3D shapes except 3-simplex, it is necessary to introduce the function $h(v_i, \sigma^n)$, the signed distance from $C(\sigma^n)$to $aff(\sigma_i^{n-1})$with the convention that $h(v_i, \sigma^n) \geq 0$when $C(\sigma^n)$and $v_i$are on the side of$aff(\sigma_i^{n-1})$. Hereσ^nbe the n-simplex, $C(\sigma^n)$ be the circumcenter of the n-simplex, $aff(\sigma_i^{n-1})$be the facet opposite to the vertex v_i of the simplex. Here in this case n=3. The magnitude of $h(v_i, \sigma^n)$can be computed as the distance between $C(\sigma^n)$and $C(\sigma_i^{n-1})$and its sign can be decided by testing whether$C(\sigma^n)$ and $v_i$have the same orientation with respect to $aff(\sigma_i^{n-1})$.

Since the pentahedron is the sum of four 3-simplex, the cost function can be defined as $f(\sigma^n) = \frac{1}{4}\sum_{j=1}^{4} \max_{v \in \sigma^n} \left|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}\right|$; where$R(\sigma^n)$ is the circumradius of n-simplex.

Remember that $-1 < \frac{h(v,\sigma^n)}{R(\sigma^n)} < 1$for finite$\sigma^n$, as $R(\sigma^n)^2 = h(v,\sigma^n)^2 + R(\sigma_i^{n-1})^2$

Theorem: The energy function (using BDAMe)$f(\sigma^n) = \frac{1}{N}\sum_{j=1}^{N} \max_{v \in \sigma^n} \left|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}\right|$, where N is the maximum total number of tetrahedron of a 3D figure, always lies between 0 and 1. The proof of this theorem is already discussed in [2].

Now we introduce another iterative method to calculate the cost function of a pyramid or quadrilateral pentahedron. Recall that the meshing of pyramid from $1.2. provided the circumcenter lies inside of the pyramid. Using that method, we get 4 tetrahedrons and 1 pyramid. If the circumcenter of this pyramid also lies inside of the pyramid, we apply the same procedure to get again 4 tetrahedrons and 1 pyramid. Following this way, we consider that the circumcenter of k-th new pyramid lies inside of the pyramid. Since all these pyramids have their own cost function and some fraction of these cost function contributes to the cost function of the given pyramid, we define the cost function as

$$f(\sigma^n) = \frac{1}{4}\sum_{j=1}^{4} \max_{v\in\sigma^n}\left|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}\right| + \frac{1}{4^k} f(\sigma^{n_k})(E1)$$

where σ^{n_k} is the n-simplex for k-th new pyramid and here the proportion for the k-th new pyramid we take is $1/4^k$. Remember that the circumcenters for all the k-th pyramid lies inside of them. We now extend the meshing described in \$2.2. (ii) for $(N+1)$-sided polyhedron with N-gonal base and having the circumcenter in itself. If we mesh in that way, we get N tetrahedrons and one $(N+1)$-sided polyhedron. Consider that the circumcenters for all the k-th iterative $(N+1)$-sided polyhedron lies inside of themselves. Then the cost function we can define as

$$f(\sigma^n) = \frac{1}{N}\sum_{j=1}^{N} \max_{v\in\sigma^n}\left|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}\right| + \frac{1}{N^k} f(\sigma^{n_k})(E2)$$

Notice that the proportion we take here is $1/N^k$ as Nis the number of tetrahedrons obtained from each one iteration. We now conclude with an important theorem.

Theorem: For infinite times of meshing of an element(object) the condition for the cost function to lie between 0 and 1 is $\alpha > \cos^{-1}(\frac{N-1}{N})$, where α is the minimum of the maximal vertex angles of the meshed tetrahedrons and Nis the number of the tetrahedrons obtained from the first meshing of the element, considering the circumcenters for all the meshed elements lie inside them.

Proof: The ratio h/R is related to the maximal vertex angle of the tetrahedron which is already described in [4]. The relation can be given by $\frac{h}{R} = \cos(\max_{i=1\ldots4}\alpha_i)(E3)$

Where the angle $\alpha_i$of vertex P_i is defined as the angle under which a circumcircle of the face opposite to P_i is seen from P_i in the plane passing through P_i and center of element's circumsphere, O and face's circucenterO_i .

Now using (E3), we get from (E2),

$$f(\sigma^n) = \frac{1}{N}(\cos\alpha_{11} + \cos\alpha_{12} + \cdots + \cos\alpha_{1N}) + \frac{1}{N^2}(\cos\alpha_{21} + \cos\alpha_{22} + \cdots + \cos\alpha_{2N}) + \cdots + \frac{1}{N^{k+1}}(\cos\alpha_{k1} + \cos\alpha_{k2} + \cdots + \cos\alpha_{kN})$$

If α_i is the minimum of the maximal vertex angle of the tetrahedrons formed from the i-th meshing i.e. $\alpha_i = \min_{j=1\ldots N}(\alpha_{ij})$ then

$$f(\sigma^n) \le \frac{1}{N}(N\cos\alpha_1) + \frac{1}{N^2}(N\cos\alpha_2) + \cdots + \frac{1}{N^{k+1}}(N\cos\alpha_k) \le \cos\alpha + \frac{1}{N}\cos\alpha + \cdots + \frac{1}{N^k}\cos\alpha$$

$$= \left(1 + \frac{1}{N} + \cdots + \frac{1}{N^k}\right)\cos\alpha$$

$$= \left(\frac{1-\frac{1}{N^k}}{1-\frac{1}{N}}\right)\cos\alpha$$

For $k \to \infty$ we get,

$$f(\sigma^n) \le \frac{N}{N-1}\cos\alpha < 1$$

$$\Rightarrow \alpha > \cos^{-1}\left(\frac{N-1}{N}\right)$$

Where α is the minimum of the maximal vertex angles. ■

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