CONTROLLING OSCILLATING BURSTING

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Abstract- We present a control method based on the entrainment control. The aim is to control oscillating bursting, generated by a class of 3D fast-slow differential equations. The basic idea is to design a control to be added to the fast variables of the system in order to drive it to a desired trajectory.

Keywords- control; Stability; neuronal activity; bursting;fast-slow systems.

I. INTRODUCTION

Many mathematical models, describing the electrical activity underlying bursting behavior, in a wide variety of cell types have been proposed. Bursting are oscillations consisting of alternating active phases of repetitive spiking and quiescence phases [3]. We are interesting in controlling models of three dimensional differential equations, with two fast variables and a single slow variable that can display multiple forms of bursting and tonic spiking behavior. There exist numerous models based on Hodgkin and Huxley neuronal model [1], to describe the membrane potential activity, for instance we can cite M. Pernarowski [2], P.Bertram and A.Sherman [4], A.Sherman and J.Rinzel [6], Hindmarsh and Rose [7]. These models of oscillating and bursting biological cells can be written or transformed into the general form:

$$\begin{cases} \dot{x} = f(x) - y - k(z), \\ \dot{y} = \mu(g(x) - y), \\ \dot{z} = \varepsilon(h(x) - z), \end{cases} \quad (1)$$

where x is the potential action, $y = \dot{x}$ and z is the slowly varying concentration, $0 < \varepsilon < 1$ and f, g, k, h are sufficiently smooth functions.

This work proves that if we add a suitable control to the fast variables of the uncontrolled system, by using the theorem of P. Seibert and R. Suarez [5] for cascade systems [10], the Liapunov and Lasalle theorems [9], the error between the trajectory of this system and the controlled one tends asymptotically to zero. Under this control, given a reference bursting oscillation, the entrainment problem consists to drive this trajectory to an arbitrary goal trajectory. This control can also suppress chaos when it is not desired.

II. ENTRAINMENT CONTROL

The system (1) can be written as a generalized Liénard equation:

$$\ddot{x} + F_1(x)\dot{x} + G_1(x) = \varepsilon \frac{dk}{dz}(z) H_1(z), \quad (2)$$

where:

$$\begin{cases} F_1(x) = -f(x) - \mu, \\ G_1(x) = \mu(g(x) - f(x)), \\ H_1(x,z) = \mu k(z) + \varepsilon(h(x) - z)\dfrac{dk}{dz}(z). \end{cases}$$

We start by the following change of variables:

$$\begin{cases} \alpha = x, \\ \beta = f(x) - y - k(z), \\ z = z, \end{cases}$$

which changes the system (1) into the form

$$\begin{cases} \dot{\alpha} = \beta, & (3) \\ \dot{\beta} = G_1(\alpha) - \beta F_1(\alpha) - H_1(\alpha, z), & (4) \\ \dot{z} = \varepsilon(h(\alpha) - z). & (5) \end{cases}$$

Let $\Gamma = (\alpha, \beta, z)$ be a given trajectory when no control is applied and let $\gamma = (\psi, \varphi, \xi)$ be the goal dynamics generated by:

$$\begin{cases} \dot{\psi} = \varphi, \\ \dot{\varphi} = G_1(\psi) - \varphi F_1(\psi) - H_1(\psi, \xi), \\ \dot{\xi} = \varepsilon(h(\psi) - \xi). \end{cases}$$

We denote by u and w the additive control term to add to the fast subsystem given by equations (3) and (4).

The control problem is to find u and w such that the system state Γ is entrained to arbitrary given γ and the error between these two trajectories tends asymptotically to zero.

Consider the error $e = (e_1, e_2, e_3)^t$ between the reference trajectory Γ and the goal one γ.

$$\begin{cases} e_1 = x - \psi, \\ e_2 = y - \varphi, \\ e_3 = z - \xi. \end{cases}$$

We propose the following control system:

$$\begin{cases} \dot{\alpha} = \beta + u(e_2, e_3), \\ \dot{\beta} = -G_1(\alpha) - \beta F_1(\alpha) - H_1(\alpha, z) + w(e_2), \quad (6) \\ \dot{z} = \varepsilon(h(\alpha) - z), \end{cases}$$

where u and w are defined as follows

$$\begin{cases} u(e_2, e_3) = e_2 + v(e_3), & (7) \\ w(e_2) = e_2(F_1(\psi) - 1). & (8) \end{cases}$$

v is an arbitrary smooth function which will be chosen below.

The equations governing the error are given by:

$$\begin{cases} \dot{e}_1 = -v(e_3), \\ \dot{e}_2 = -G(e_1) - e_2 F(e_1) - H(e_1, e_3) + e_2(F_1(\psi) - 1), \quad (9) \\ \dot{e}_3 = \varepsilon K(e_1, e_3), \end{cases}$$

Where

$$\begin{cases} G(e_1) = (G_1(e_1 + \psi) - G_1(\psi)) + \varphi(F_1(e_1 + \psi) - F_1(\psi)), \\ F(e_1) = F_1(e_1 + \psi), \\ H(e_1, e_3) = H_1(e_1 + \psi, e_3 + \xi) - H_1(\psi, \xi), \\ K(e_1, e_3) = (h(e_1 + \psi) - h(\psi)) - e_3. \end{cases}$$

Choosing $v(0) = 0$ shows that $e = 0$ is an equilibrium point of the system (9).

The goal is then to prove the asymptotic stability of this equilibrium point for the error system.
In what follows, we prove that the entrainment of the system (1) to an arbitrary goal can be achieved starting from any initial condition.

Proposition
Let the origin be an equilibrium of the system (9).
Let u and w be the additive control terms defined by (7) and (8).
Suppose the function q defined by

$$q(x) = h(x + \psi) - h(\psi) \qquad (10)$$

satisfies the following properties:

$$q \in C^1(R), \; q(0) = 0, \qquad (11)$$

and

$$xq(x) > 0, \quad \forall x \neq 0, \qquad (12)$$

then the system is asymptotically stable.

Proof:
We first use the following theorem for cascade systems:

Theorem (P. Seibert and R. Suarez)
Consider the cascade system

$$\begin{cases} \dot{x} = f_1(x, y), & (13) \\ \dot{y} = f_2(y), & (14) \end{cases}$$

where

$$f_1 : R^{n_1} \times R^{n_2} \to R^{n_1},$$
$$f_2 : R^{n_2} \to R^{n_2}$$

are locally Lipschitz in (x, y).
Suppose both

$$\dot{x} = f_1(x, 0) \qquad (15)$$

and equation (14) have globally asymptotically stable equilibrium at their respective origin, then the origin of the cascade system (13),(14) is globally asymptotically stable.

The system (6), can be written as the following cascade system:

$$\begin{cases} \dot{e}_2 = f_1(e_1, e_2, e_3), & (16) \\ \dot{E} = f_2(E), & (17) \end{cases}$$

where $E = (e_1, e_3)$, and

$$\begin{cases} f_1(e_1, e_2, e_3) = -G(e_1) - e_2 F(e_1) - H(e_1, e_3) + e_2(F_1(\psi) - 1), \\ f_2(E) = (-v(e_1), \varepsilon K(e_1, e_3)). \end{cases}$$

The differential equation $\dot{e}_2 = f_1(0, e_2, 0) = -e_2$, is globally asymptotically stable.
Indeed, if we choose the Lyapunov candidate function such that

$$V(e_2) = \frac{1}{2} {e_2}^2,$$

then V positive definite and

$$\dot{V}(e_2) = -{e_2}^2 \leq 0,$$

Therefore V is a Lyapunov function and $e_2 = 0$ is globally asymptotically stable.

Now consider the system

$$\dot{E} = f_2(E)$$

and consider the candidate Lyapunov function

$$V(e_1, e_3) = \varepsilon \int_0^{e_1} q(s) ds + \frac{1}{2} {e_3}^2,$$

with q defined in (10).

Since q verifies property (11) and (12), V is positive definite.

The derivative of V along trajectories is given by:

$$\dot{V}(e_1,e_3)=\varepsilon\dot{e}_1 q(e_1)+e_3\dot{e}_3$$
$$=-\varepsilon v(e_3)q(e_1)+\varepsilon e_3(q(e_1)-e_3).$$

By choosing $v(e_3)=e_3$, we obtain

$$\dot{V}(e_1,e_3)=-\varepsilon.e_3{}^2\le 0.$$

Using Lasalle theorem shows that $(e_1,e_3)=(0,0)$ is globally asymptotically stable for the system (17).

Hence the origin is asymptotically stable for the system (9) . Therefore the controlled trajectory tends asymptotically to the goal trajectory γ.

III. NUMERICAL RESULTS

To illustrate the effect of the control, we apply the proposed control (7), (8) to the following model,

A. The β-cells model

Insulin-secreting β-cells, located within the pancreatic islets of Langerhans, are excitable cells that produce regular bursts of action potentials when stimulated by glucose. For more details see [2].

The model takes the form

$$\begin{cases}\dot{x}=f(x)-y-k(z),\\ \dot{y}=\dfrac{1}{\tau}(w_\infty(x)-y),\\ \dot{z}=\varepsilon(h(x)-z),\end{cases}$$

where

$$f(x)=-\frac{a}{3}x^3+a\mu x^2+(\frac{1}{\tau}-a(\mu^2-\eta^2))x,$$

$$w_\infty(x)=(\tau-\frac{a}{3})x^3+a\mu x^2+(\frac{1}{\tau}-a(\mu^2-\eta^2)-3\tau)x$$
$$+(\frac{1}{\tau}-a(\mu^2-\eta^2)-3\tau),$$

$$k(z)=\tau z,$$

$$h(x)=x-b.$$

The values of the parameters are given in the following table:

Parameter	Value
a	0.25
β	4
τ	1
b	-0.954

The parameters μ and η will be varied in order to obtain different trajectories of this system.

For this system $q(x)$ verifies the conditions (11) and (12). In order to illustrate the effect of the driving term, we fix the parameter ε to $\varepsilon=0.0025$.

We depict, in respectively Figure1 and Figure2, the bursting reference trajectory Γ for $\mu=2.25,\eta=0.5$ and the goal bursting one for $\mu=1.5,\eta=0.7$.

Figure3 shows the asymptotic driving of the Γ trajectory to the γ trajectory when we add the control to the β-cells model. The error between the referenced trajectory and the driven one are depicted in figure 4 showing then its asymptotic convergence.

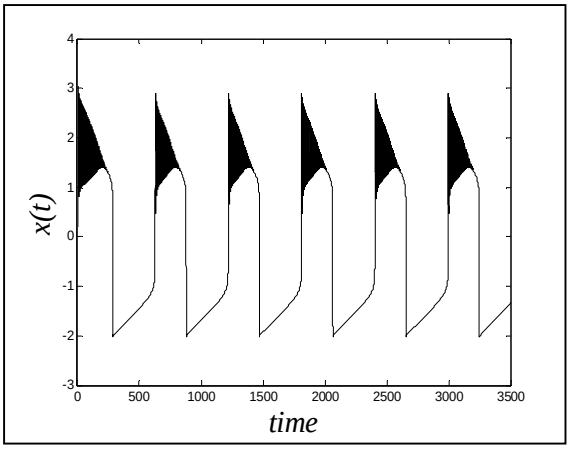


Fig. 1 Bursting oscillations Γ of membrane potential for $\mu=2.25,\eta=0.5$ and initial conditions (0.2,0.4,0.0001)

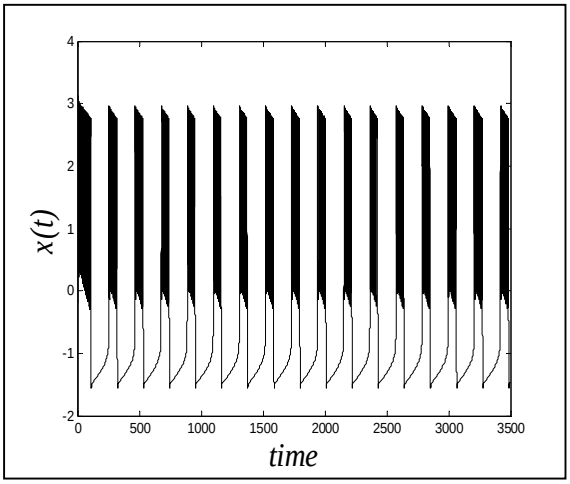


Fig. 2 Bursting oscillations γ of membrane potential for $\mu=1.5,\eta=0.7$ and initial conditions (-0.2,0.1,0.0001)

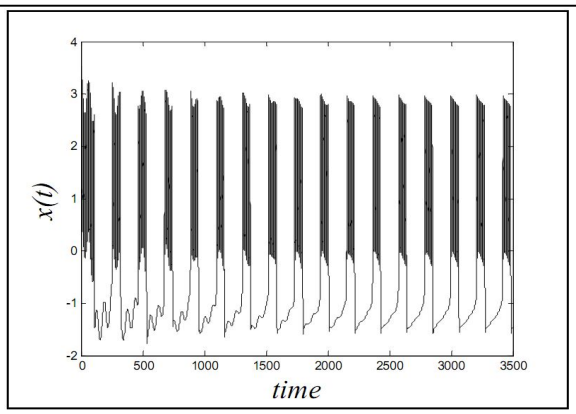


Fig. 3 Driven trajectory from Γ to γ by adding the control.

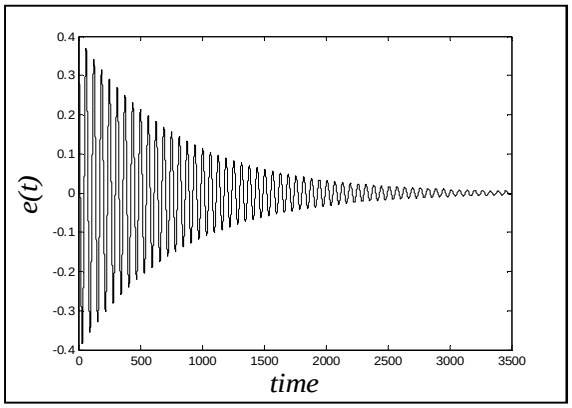


Fig. 4 Error between the controlled trajectory and Γ.

REFERENCES

[1] A.L. Hodgkin and A.F. Huxley, A quantitative description of membrane current and its application to contribution and excitation in nerve, J. Physiol. (London) 117 (1952) 500-544

[2] M. Pernarowski, Fast and slow subsystems for a continuum model of bursting activity in the pancreatic islet, SIAM J. Appl. Math., 58 (1998), pp. 1667–1687.

[3] E. Izhikevich, Neural excitability, spiking, and bursting, Inter. J. Bifur. Chaos Appl. Sci.Engrg., 10 (2000), pp. 1171–1266.

[4] Richard Bertram Arthur Sherman, A Calcium-based Phantom Bursting Model for Pancreatic Islets, Bulletin of Mathematical Biology (2004) 66, 1313–1344

[5] P. Seibert and R. Suarez, On the problem of stability in the presence of an invariant manifold. Application to stabilizability of nonlinear control systems, Reporte de Investigacion UAM-I, Vol. VI, No. 4 (1988), Mexico,D.F., Mexico

[6] A. Sherman and J. Rinzel, Model for synchronization of pancreatic β-cells by gap junctions,Biophys. J., 59 (1991), pp. 547–559.

[7] J.L. Hindmarsh and R.M. Rose, A model of neuronal bursting using three coupled first order differential equations.Proc.R.Soc.B,221, 87-102.

[8] A E. Milne and Z.S. Chalabi, Control analysis of the Rose-Hindmarsh model for neural activity,IMA Journal of Mathematics Applied in Medicine and Biology (2001) 18, 53-75

[9] Lasalle J.P. Some extensions of Lyapunov'method. IRE Trans. Circuit Theory (1960) CT 7. 520-527.

[10] H. K. Khalil, Nonlinear Systems, 3rd ed. Prentice Hall, 2002.

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