11'S AS A MULTIPLIER

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Abstract - Technology has benefitted human lives a lot by making it better, easy, and possible. But, it is also a product of human from his creative and critical thinking in the form of the applications of equations. This illustrates how important is to discover those equations and its uses in real – life situations. This qualitative study was investigated to find a formula that serves as a shortcut when multiplying the 11's with a positive integer. It was inspired and developed from the divisibility rules in arithmetic taught in elementary. It was initiated when a number multiplied by 11 has a product which is the just the sum of the succeeding digits of the multiplicand. To extend this investigation, some positive integers were multiplied with 111, 1111, and so on. This leads to the theorem that explains the formula for multiplying positive integer N with 111…1; that is, N x 111…1 is equal to $\sum_{i=1}^{k+r-1} 10^{i-1}(n_i + n_{i-1} + n_{i-2} + \cdots + n_{i-r+1} + x_{i-1} - 10x_i)$ for some positive integers N, n, i, k, x, and r. This study further concluded that formulating equations is what it makes a critical thinker wise and what it makes technology even more advanced every now and then.

Keywords - Multiplication With11's and Divisibility Rules

I. INTRODUCTION

Discovery of the concepts and formulating its equation is the most essential work of a researcher. It makes many things possible. It is where the development of technology started. It also shows that our daily living and everything surrounding us are played by the equations of numbers like how the heart beats; when the earthquakes occur; the sum of the sequence of numbers is can be computed by $\frac{n(n+1)}{2}$; divisibility rules give shortcuts of determining whether a number is divisible by another number, and so on.

These techniques in divisibility rules encourage students to do more exploration of positive numbers just like how the products of 11's and some positive integers are formulated. Four lemmas are developed to arrive to the main theorem of this study. It is anticipated that this study motivates anyone to do more explorations of numbers and make equations out of it.

II. STATEMENT OF THE PROBLEM

This study is investigated to establish a formula for the products of N and k digits of 11 where N and k are any positive integers greater than 1.

III. CONCEPTUAL FRAMEWORK

Divisibility rules improve skills of students in determining whether a number is divisible by a particular positive integer instead of dividing it. For instance, an even number is always divisible by 2; a number whose sum of its digits is a multiple of 3 is divisible by 3; a number whose difference of the sums of the alternating digits is a multiple of 11 is divisible by 11; and so on. This is where and how this study developed when a number multiplied by 11 resulted to the product whose digits are sum of each succeeding digits. To extend further, exploration of the shortcuts for the products of positive integers with 111, 1111, and so on are done. These products have digits which are sum of the sequence of the digits of the multiplier depend on both the digits of multiplier and multiplicand which leads to the four lemmas and one theorem. In order to have fast in multiplying N and 11's, a mastery of the formula should be done.
In elementary mathematics, divisibility rules provide shortcuts in determining whether a numbers is divisible by another number. Thus, it is very essential in the learning of students.

IV. THEORETICAL FRAMEWORK

This study applied the positive integral exponents, distribution of multiplication over addition, grouping of terms, and summation property (Coronel, 2013; Leithold, 1992; Swokowski& Mendoza, 2002; Vance, 1975). Positive integral exponents refer to the expansion of a real number a with positive integer n as a power, that is, a^n equals a•a•a•…•a (n factors of a). This idea was applied in the powers of tens involved in the theorem. Also, distribution of multiplication over addition as a property of real numbers was used in the Lemmas in developing the equations. It is defined as for any three real numbers a, b, and c, a (b + c) equals ab + ac.
Similarly, grouping of terms and summation property for double summation are used in solving the equations of the Lemmas and the Theorem. Whereas, grouping of terms is applicable only to terms in such a way that the other types of factoring can be applied. Another theorem in Algebra utilized is the double summation of an expression. It was used in the interchanging of the values in Lemma 4. That is, $\sum_{i=1}^{k}\sum_{j=1}^{r} n_{ij}$ is equal to $\sum_{j=1}^{r}\sum_{i1}^{k} n_{ij}$ for some real number n and positive integers i, j, k, and r (Leithold, 1992).

V. DESIGN, SAMPLING TECHNIQUES, AND PROCEDURE

This research is a qualitative in design particularly a content analysis on the concepts of exploring the products of 11's with positive integers to develop an equation that serves as a shortcut in manual multiplication. It is done through convenient sampling of the available resources such as textbooks and other documents that served as the participants.
It started with the multiplication of numbers with 11 and developed an equation. This was followed by multiplying the same selected positive integers with 111 and again developed another equation. Then, multiplying the same selected positive integers by 1111 until 111…1 with r digits of 11's for some positive integer r. This leads to the sequence of the results of the developed equations, which are actually products of 11's and some integers, as discussed in the next section.

VI. DISCUSSION OF THE RESULTS

After the exploration of the products of some integers with 11's, four lemmas and one theorem are developed. Lemma 1 explained the product of integer N with k digits and 11 for some positive integers N and k.
Lemma 1. Let N and k be positive integers such that $N = n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0$. Then, $N \times 11 = \sum_{i=1}^{k+2} 10^{i-1}(n_i + n_{i-1} + x_i - y_i)$ for some positive integer k, $x_i = 1$ when $n_{i-1} + n_{i-2} + x_{i-1} \geq 10$, and $y_i = 10$ when $n_i + n_{i-1} + x_i \geq 10$. Otherwise, $x_i = y_i = 0$.
Proof: Suppose N is a positive integer such that $N = n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0$. Then,

$N \times 11 = (n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0) \times 11$
$= (n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0) \times (10 + 1)$
$= (n_k\,10^{k} + n_{k-1}\,10^{k-1} + n_{k-2}\,10^{k-2} + \dots + n_2\,10^2 + n_1\,10^1) + (n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0)$
$= n_k\,10^k + n_k\,10^{k-1} + n_{k-1}\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-2} + n_{k-2}\,10^{k-3+\dots+} n_2\,10^2 + n_2\,10^1 + n_1\,10^1 + + n_1\,10^0$
$= n_k\,10^k + 10^{k-1}(n_k + n_{k-1}) + 10^{k-2}(n_{k-1} + n_{k-2}) + 10^{k-3}(n_{k-2} + n_{k-3}) \cdots + 10^2(n_3 + n_2)\,10^1(n_2 + n_1) + n_1\,10^0$

Some of the sums of n_j's maybe greater than or equal to 10.
Let x and y be positive integers such that x_i's be a carried number on every sum of n_j's, j = 1, 2, …, i - 1 and previous carried number x_{i-1}. If $n_{i-1} + n_{i-2} + x_{i-1} \geq 10$ for some integer i, then $x_i = 1$ and $y_i = 10$. Otherwise, $x_i = y_i = 0$. So, adjustments can be done by adding x_i and $-y_i$ to each of the terms in the above equation and its value is still the same. That is,

$N \times 11 = 10^k(n_k + x_{k+1} - y_{k+1}) + 10^{k-1}(n_k + n_{k-1} + x_k - y_k) + 10^{k-2}(n_{k-1} + n_{k-2} + x_{k-1} - y_{k-1}) + 10^{k-3}(n_{k-2} + n_{k-3} + x_{k-2} - y_{k-2}) \cdots + 10^2(n_3 + n_2 + x_3 - y_3) + 10^1(n_2 + n_1 + x_2 - y_2) + 10^0(n_1 + x_1 - y_1)$
$= \sum_{i=1}^{k+2} 10^{i-1}(n_i + n_{i-1} + x_i - y_i)$

Conversely, expanding the summation in the right side and removing the adjustments give the two factors N and 11.

Examples: Find the product.
1. 345 x 11

Solution: Let $n_1 = 5$, $n_2 = 4$, and $n_3 = 3$ with k = 3. Then using Lemma 1,

$345 \times 11 = \sum_{i=1}^{5} 10^{i-1}(n_i + n_{i-1} + x_i - y_i)$
$= 10^0(n_1 + n_0 + x_1 - y_1) + 10^1(n_2 + n_1 + x_2 - y_2) + 10^2(n_3 + n_2 + x_3 - y_3) + 10^3(n_4 + n_3 + x_4 - y_4) + 10^4(n_5 + n_4 + x_5 - y_5)$
$= 1(5 + 0 + 0 - 0) + 10^1(4 + 5 + 0 - 0) + 10^2(3 + 4 + 0 - 0) + 10^3(0 + 3 + 0 - 0) + 10^4(0 + 0 + 0 - 0)$
$= 3795.$

2. 87658 x 11

Solution: Note that $n_1 = 8$, $n_2 = 5$, $n_3 = 6$, $n_4 = 7$, and $n_5 = 8$ with k = 5. Then,

$87658 \times 11 = \sum_{i=1}^{7} 10^{i-1}(n_i + n_{i-1} + x_i - y_i)$
$= 10^0(n_1 + n_0 + x_1 - y_1) + 10^1(n_2 + n_1 + x_2 - y_2) + 10^2(n_3 + n_2 + x_3 - y_3) + 10^3(n_4 + n_3 + x_4 - y_4) + 10^4(n_5 + n_4 + x_5 - y_5) + 10^5(n_6 + n_5 + x_6 - y_6) + 10^6(n_7 + n_6 + x_7 - y_7)$
$= 10^0(8 + 0 + 0 - 0) + 10^1(5 + 8 + 0 - 10) + 10^2(6 + 5 + 1 - 10) + 10^3(7 + 6 + 1 - 10) + 10^4(8 + 7 + 1 - 10) + 10^5(0 + 8 + 1 - 0) + 10^6(0 + 0 + 0 - 0)$
$= 1(8) + 10(3) + 100(2) + 1000(4) + 10000(6) + 100000(9) + 0$
$= 964, 238.$

To extend the product of N and 11's, Lemma 2 was developed for the product of N and 111.
Lemma 2. Let N and M be positive integers such that $N = n_k\,10^{k-1} + n_{k-1}\,10^{k-2} + n_{k-2}\,10^{k-3} + \dots + n_2\,10^1 + n_1\,10^0$ with k digits of N and $M_i = n_i + n_{i-1} + n_{i-2} + r_i - 10r_{i+1}$ for some positive integers, n, i, and r where r_i's are carried numbers. Then, $N \times 111 = \sum_{i=1}^{k+3} 10^{i-1} M_i$.
Proof: (Use Lemma 1 to prove this theorem.)
Examples: Evaluate
3. 345 x 111

Solution:
$345 \times 111 = \sum_{i=1}^{k+3} 10^{i-1} M_i$
$= \sum_{i=1}^{6} 10^{i-1}(n_i + n_{i-1} + n_{i-2} + r_i - 10r_{i+1})$

$= 10^0(n_1 + n_0 + n_{-1} + r_1 - 10r_2) + 10^1(n_2 + n_1 + n_0 + r_2 - 10r_3) + 10^2(n_3 + n_2 + n_1 + r_3 - 10r_4) + 10^3(n_4 + n_3 + n_2 + r_4 - 10r_5) + 10^4(n_5 + n_4 + n_3 + r_5 - 10r_6) + 10^5(n_6 + n_5 + n_4 + r_6 - 10r_6)$

$= 10^0(5 + 0 + 0 + 0 - 0) + 10^1(4 + 5 + 0 + 0 - 0) + 10^2(3 + 4 + 5 + 0 - 10) + 10^3(0 + 3 + 4 + 1 - 0) + 10^4(0 + 0 + 3 + 0 - 0) + 10^5(0 + 0 + 0 + 0 - 0)$

$= 38,295$

4. 87658 x 111 = $\sum_{i=1}^{8} 10^{i-1} M_i$

$= 10^0(n_1 + n_0 + n_{-1} + r_1 - 10r_2) + 10^1(n_2 + n_1 + n_0 + r_2 - 10r_3) + 10^2(n_3 + n_2 + n_1 + r_3 - 10r_4) + 10^3(n_4 + n_3 + n_2 + r_4 - 10r_5) + 10^4(n_5 + n_4 + n_3 + r_5 - 10r_6) + 10^5(n_6 + n_5 + n_4 + r_6 - 10r_7) + 10^6(n_7 + n_6 + n_5 + r_6 - 10r_7) + 10^7(n_7 + n_6 + n_5 + r_7 - 10r_8)$

$= 10^0(8 + 0 + 0 + 0 - 0) + 10^1(5 + 8 + 0 + 0 - 10) + 10^2(6 + 5 + 8 + 1 - 20) + 10^3(7 + 6 + 5 + 2 - 20) + 10^4(8 + 7 + 6 + 2 - 20) + 10^5(0 + 8 + 7 + 2 - 10) + 10^6(0 + 0 + 8 + 1 - 0) + 10^7(0 + 0 + 0 + 0 - 0)$

= 9, 730, 038.

The results in Lemma 1 and Lemma 2 lead to investigate thoroughly the products of integer N with 11's. This time, the product of the four digits of 11 and N gives different equation as discussed in the next lemma.

Lemma 3. Let N and M be positive integers with $N = n_k 10^{k-1} + n_{k-1} 10^{k-2} + n_{k-2} 10^{k-3} + \dots + n_2 10^1 + n_1 10^0$ and $M_i = n_i + n_{i-1} + n_{i-2} + n_{i-3} + r_i - 10r_{i+1}$ for some integers n, i, and r. Then, N x 1111 = $\sum_{i=1}^{k+4} 10^{i-1} M_i$.

Proof: (The same proof of Lemma 2 is applied here.)

Example: Find the product.

5. 6789 x 1111

Solution: We have k = 4, $n_1 = 9$, $n_2 = 8$, $n_3 = 7$, and $n_4 = 6$. Note that other n_i'sarezeros.

Then,

6789 x 1111 = $\sum_{i=1}^{k+4} 10^{i-1} M_i$

$= \sum_{i=1}^{7} 10^{i-1}(n_i + n_{i-1} + n_{i-2} + + n_{i-3} + r_i - 10r_{i+1})$

$= 10^0(n_1 + n_0 + n_{-1} + n_{-2} + r_1 - 10r_2) + 10^1(n_2 + n_1 + n_{-1} + n_{-2} + r_2 - 10r_3) + 10^2(n_3 + n_2 + n_1 + n_0 + r_3 - 10r_4) + 10^3(n_4 + n_3 + n_2 + n_1 + r_4 - 10r_5) + 10^4(n_5 + n_4 + n_3 + n_2 + r_5 - 10r_6) + 10^5(n_6 + n_5 + n_4 + n_3 + r_6 - 10r_7) + 10^6(n_7 + n_6 + n_5 + n_4 + r_7 - 10r_8) + 10^7(n_8 + n_7 + n_6 + n_5 + r_7 - 10r_8)$

= 1(9+0 + 0 + 0+ 0 – 0) + 10(8 + 9 + 0 + 0 + 0 – 10) + 10^2(7 + 8 + 9 + 0 + 1 – 20) + 10^3(6 + 7 + 8 + 9 + 2 – 30) + 10^4(0 + 6+7 + 8 + 3 – 20) + 10^5(0 + 0 +6 +7 + 2 – 10) + 10^6(0 + 0 + 0 +6 + 1 – 0) + 10^7(0 + 0 + 0 + 0 – 0)

= 7, 542, 579.

Previous lemmas provide equations with interesting characteristics because they are products of N and 11's which are sequence with respect to the digits of both factors. Theorem 4 shows the generalization of the concepts.

Theorem 4. SupposeN and M be positive integers such that $N = n_k 10^{k-1} + n_{k-1} 10^{k-2} + n_{k-2} 10^{k-3} + \dots + n_2 10^1 + n_1 10^0$ and $M_i = n_i + n_{i-1} + n_{i-2} + \cdots + n_{i-j+1} + r_i - 10 r_{i+1}$ for some integers n, r, k, i, and j.If 11's has j digits then, N x 111 ... 1 = $\sum_{i=1}^{k+j} 10^{i-1} M_i$.

Proof: The same proof of Lemma1 can be applied here.

Example:

6. 98776 x 111

Solution: Given are $n_1 = 6$, $n_2 = 7$, $n_3 = 7$, $n_4 = 8$, $n_5 = 9$ with k = 5 and j = 3. Then,

98776 x 111 = $\sum_{i=1}^{k+j} 10^{i-1} M_i$

$= \sum_{i=1}^{k+j} 10^{i-1}(n_i + n_{i-1} + \cdots + n_{i-j+1} + r_i - 10r_{i+1})$

$= 10^0(n_1 + r_1 - 10r_2) + 10^1(n_1 + n_2 + r_2 - 10r_3) + 10^2(n_1 + n_2 + n_3 + r_3 - 10r_4) + 10^3(n_2 + n_3 + n_4 + r_4 - 10r_5) + 10^4(n_3 + n_4 + n_5 + r_6 - 10r_7) + 10^5(n_4 + n_5 + n_6 + r_7 - 10r_8) + 10^6(n_5 + n_6 + n_7 + r_7 - 10r_8) + 10^7(n_6 + n_7 + n_8 + r_8 - 10r_9)$

$= 10^0(6 + 0 - 10(0)) + 10^1(6 + 7 + 0 - 10(1)) + 10^2(6 + 7 + 7 + 1 - 10(2)) + 10^3(7 + 7 + 8 + 2 - 10(2)) + 10^4(7 + 8 + 9 + 2 - 10(2)) + 10^5(8 + 9 + 0 + 2 - 10(1)) + 10^6(9 + 0 + 0 + 1 - 10(1)) + 10^7(0 + 0 + 0 + 0 - 0)$

= 10, 964, 136

This concept in Theorem 4 can be illustrated in Table 1 as follows.

Table 1. Illustration of Theorem 4

Carried Number (r_i)	r_{k+j-1}	…	R_3	R_2	R_1
Number	n_{k+j-1}	…	n_3	n_2	n_1
Product	M_{k+j-1}	…	M_3	M_2	M_1
Place of Value	10^{k+j-1}	…	10^3	10^1	10^0

Theorem 5 shows another formula for exploring the products of N and 11's.

Theorem 5. Let N be a positive integer with k digits such that $N = n_k 10^{k-1} + n_{k-1} 10^{k-2} + n_{k-2} 10^{k-3} + \dots + n_2 10^1 + n_1 10^0$ for some integer n. If 11's has r digits then N x 1111…1 = $\sum_{i=1}^{k} \sum_{j=1}^{r} 10^{i+j-2} n_i$ for some positive integers i, j, and r.

Proof: Let $N = n_k\ 10^{k-1} + n_{k-1}\ 10^{k-2} + n_{k-2}\ 10^{k-3} + \dots + n_2\ 10^1 + n_1\ 10^0$

$N \times 1111\dots = [n_k\ 10^{k-1} + n_{k-1}\ 10^{k-2} + n_{k-2}\ 10^{k-3} + \dots + n_2\ 10^1 + n_1\ 10^0] \times [10^{r-1} + 10^{r-2} + \dots + 10 + 10^0]$

$= (\sum_{i=1}^{k} n_i 10^{i-1}) \times (\sum_{j=1}^{r} 10^{j-1})$

$= \sum_{j=1}^{r}(\sum_{i=1}^{k} 10^{i+j-2} n_i)$

$= \sum_{i=1}^{k}(\sum_{j=1}^{r} 10^{i+j-2} n_i)$.

Conversely, the expansion of the two summations at the right side leads to the factors N and 111…11.

Examples:

7. 98776 x 111

Solution:

$98776 \times 111 = \sum_{i=1}^{k}\sum_{j=1}^{r} 10^{i+j-2} n_i$

$= \sum_{i=1}^{5}\sum_{j=1}^{3} 10^{i+j-2} n_i$

$= \sum_{i=1}^{5}(10^{i-1} n_i + 10^{i} n_i + 10^{i+1} n_i)$

$= \sum_{i=1}^{5} 10^{i-1} n_i + \sum_{i=1}^{5} 10^{i} n_i + \sum_{i=1}^{5} 10^{i+1} n_i$

$= (10^0\ n_1 + 10 n_2 + 10^2 n_3 + 10^3 n_4 + 10^4 n_5) + (10^1\ n_1 + 10^2\ n_2 + 10^3 n_3 + 10^4 n_4 + 10^5 n_5) + (10^2\ n_1 + 10^3\ n_2 + 10^4 n_3 + 10^5 n_4 + 10^6 n_5)$

$= [1(6)+10(7)+10^2(7)+10^3(8)+10^4(9)] + [10(6)+10^2(7)+10^3(7) + 10^4(8) + 10^5(9)] + [10^2(6)+10^3(7)+10^4(7)+10^5(8)+10^6(9)]$

$= 98776 + 987760 + 9877600$

$= 10, 964, 136$

CONCLUSION

This study has shown some equations derived from the concept of multiplying any numbers by 11's. It shows not only techniques that attract students in learning the concept but explorations of numbers that anyone can do. The results also generalized those ideas shown by many mathematics teachers along with the idea formulated here (Solaiman, 2017). This study does not limit the concept of exploring more about the multiplications of integers and motivates the readers to discover more concepts.

IMPLICATIONS

Techniques in multiplication inspire and stimulate learners to learn it. This initiates their liking of the concept. When it puts into equation, students not only amazed from the one who made it but eventually start of thinking if he or she could do the same in the future through formulating equations.

Thus, this study has advantages in developing the student's higher order of thinking skills and thus recommends for incorporating it in mathematics curriculum.

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