RANK OF SEMIGROUP: GENERATING SETS AND INDEPENDENT

[1]ZSOLT LIPCSEY, [2]SAMPSON MARSHAL IMEH

Department of Mathematics, University of Calabar, Calabar, C.R.S., Nigeria.
Department of Mathematics and Statistics, AkwaIbom State University, IkotAkpaden.
E-mail: marshalsampson@aksu.edu.ng, lipzsoltt@gmail.com

Abstract - We investigate a certain characterisation for rank of a semigroup by Howie and Ribeiro (1999), to ascertain the relevance of the concept of independence. There are cases where the concept of independence fails to be useful for this purpose. One would expect the basis element to be the maximal independent subset of a given semigroup. However, we construct examples for semigroups - finite and infinite, commutative and non-commutative, where there exists finite basis and the number of independent elements is larger than the basis.

Index Terms - Generating sets, Independent Set, Rank, Cyclic Semigroup, Basis, Commutative.

I. INTRODUCTION

An algebraic structure has many elements and it is practical to have a small subsystem of the structure which facilitates the construction of all the elements using the operation(s) of the structure. Such subsystems are called generator systems. Our interest is to study such small subsystems especially as it concerns their sizes. In particular, the size k of the smallest subset Uof S, in which every element of S can be generated as the product of the elements of U is what we call the rank of the semigroup.

The notion of 'rank' or 'dimension' belongs primarily to linear algebra. In linear Algebra, Rank can be defined for a finite vector space V either as the size of a maximal linear independent subset or as the size of a minimal generating set of V, and it is an elementary result in linear algebra that these two cardinalities are equal.

However, in the case of a more general algebraic system like semigroup, the possible definitions of rank are not just in terms of maximal linear independent subsets or as the size of a minimal generating set because results show that both do not coincide in some cases like in the case of Vector Spaces. One example is the Brandt semigroup in which the size of maximal independent subset is different from the size of its minimal generating set.
Consequent upon this fact that the rank of a semigroup in terms of minimal generating set fails to coincide, in some cases, with the rank in terms of maximal independent set, Howie and Ribeiro (1999) gave a definition for five (5) different types of rank (as we shall later see) to sufficiently characterise rank for all possible semigroups. This definition accommodates the idea of minimal generating set and that of maximal independent subset. The five ranks given in the definition do not all give the same value so that it is possible to have varying values as rank for a particular semigroup depending on which definition we choose to use.

1.1. **Definition**
A group is a set G together with a binary operation * such that
(i) $\exists e \in G$ such that $e * x = x = x * e, x \in G$
(ii) for any $x \in G, \exists x^{\#} \in G$ such that $x * x^{\#} = x^{\#} * x = e \in G$
(iii) for $x, y, z \in G. x * (y * z) = (x * y) * z$

1.2. **Definition**
A semigroup is a set S together with a binary operation * (that is a function $*: S \times S \to S$) that satisfies the associative property: $\forall a, b, c \in S$ such that $a * (b * c) = (a * b) * c$ holds.

1.3. **Definition (Generating sets)**
Let X be a subset of a semigroup S, X generates S as a semigroup if every element of S can be written as a product of elements of X.

1.4. **Remark**
The concept of independence in semigroup is analogous to that of independence in vector space. We state below the definition of independence in vector space and then give the analogous definition in semigroup.

1.5. **Definition (Linearly Dependent Sets)**
The set $\{V_i\}, i = 1,2, \dots, n$, where V_i are vectors, is said to be linearly dependent if there exist scalars $k_i = 1,2, \dots, n, k_i$ not all zeros, such that

$$\sum_{i=1}^{n} k_i v_i = 0$$

1.6. **Definition (Linearly Independent Sets)**
The set $\{V_i\}, i = 1,2, \dots, n$, where V_i are vectors, is said to be linearly independent if there exists scalars k_i, $i = 1,2, \dots, n$, such that

$$\sum_{i=1}^{n} k_i v_i = 0$$

Implies $k_i = 0$, for all $i = 1,2, \dots, n$.

1.7. **Definition** (Maximal Independent Set of vector Spaces)
Let U be a set of linearly independent vectors u_i for $1 \leq i \leq n$, and let $u^* \notin U$ be any vector. Uis a maximal independent set if $U \cup \{u^*\}$ is dependent.

1.8. **Remarks**
A Maximal linearly independent set of vectors is that in which including any other vector in the vector space would make it linearly dependent. (i.e., if any other vectorin the space can be expressed as a linear combination of elements of a maximal set--the basis). There can be more than one maximal independent subsets of a set, but they must each contain the same number of elements. We now consider a concrete example for the concept of linear independence that the concept being studied in semigroup mimics.

1.9. **Example** (Maximal Independent Subsets)
In R_5, we wish to find a maximal Independent set of the set $(A, *)$ defined by
$A =$
$\{(1,2,0,-2,1), (3,0,-4,4,3), (0,3,2,-5,0), (0,0,0,1,2), (2,1,-2,3,6)\}$
where * is as used in vectors multiplication.
Solution.
Let $A = \{u_i\}, for\ i = 1,2, \ldots 5.$
Since $u_1 \neq 0$, the subset $\{u_1\}$ is independent. Enlarging this subset by adding u_2 to it, we find that the subset $\{u_i\}, for\ i = 1,2$ is still independent because for scalars $k_1\ and\ k_2$, $\sum_{k=1}^{2} k_i u_i = 0$ implies $k_1 = k_2 = 0$. Enlarging again by adding u_3, we find that $\{u_i\}$, $i = 1,2,3$, is dependent because we cannot find scalars $k_i, i = 1,2,3$ not all zeros so that $\sum_{k=1}^{3} k_i u_i = 0$. So we discard u_3 and try $u_4$instead. The set $\{u_i\}, for\ i = 1,2,4$, is independent; hence we move on to enlarge the set by bringing in u_5. It will be observed that
$\{u_i\}, for\ i = 1,2,4,5$,is dependent hence we discard u_5. The set
$B = \{u_i\}\ for\ i = 1,2,4$ is a maximal independent subset of A.

Now notice that
$B = \{u_1, u_2, u_{4,}\}$
is not the only maximal independent subset. The reason is that if we take the elements in a different order say $u_5, u_4, u_3, u_2,\ u_1$, we will have the maximal Independent subset as $C = \{u_5, , u_4, , u_1\}$and not $B = \{u_1, u_2, u_{4,}\}$. Somebody else may decide to use the order
$u_2, u_4, u_3, u_1, u_5$to obtain a different maximal independent subset.
There are $n!$Possible maximal independent subsets (where$n = 5$ in this case)so that we have 120 possible number of maximal independent subsets.
Notice also that the sets B and C have the same number of elements. This is the case for all Maximal Independent subsets of any vector space. All maximal independent subsets have the same number of elements and that number is what we call the dimension of A and also referred to as the generator of A given as $\langle A \rangle$.
For finite sets, $\langle A \rangle = A$.
Each maximal independent subset is called a basis of $\langle A \rangle$. So B and C are basis.

1.10. **Remark**
The definition of 1.6 and 1.7 as well as the example of section 1.9, specifically apply to vector spaces. However, semigroups are not vector spaces and are defined in terms of a single operation also called multiplication. In our work on semigroup, we will utilise an analogous definition of Independence of sets as given below.

1.11. **Definition (Independent Sets of semigroup)**
Let S be a finite semigroup. A subset U of S is independent if, for every $u$in U, the element u does not belong to the subsemigroup $\langle U \backslash \{u\} \rangle$ generated by the remaining elements of U.

1.12. **Definition** (Rank of a semigroup)
The rank $r(s)$ of a semigroup$S$ is the size of the minimal generating set $U \subseteq S$.

1.13. **Example**
Consider the six elements semigroup$S$, which is a 2×2 real matrix under matrix multiplication,

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplication table:

	o	**i**	**a**	**b**	**ab**	**ba**
o	o	o	o	o	o	o
i	o	i	a	b	ab	ba
a	o	a	o	ab	o	a
b	o	b	ba	o	b	o
ab	o	ab	a	o	ab	o
ba	b	ba	o	b	o	ba

Let $U = \{i, a, b\} \subseteq S$. We show that U is a maximal independent subset:
If we take out $i$from U, notice that $\{i\}$ is excluded from the subsemigroup generated by the remaining elements $U \setminus \{i\}$ of S. That is, $\{i\}$ is excluded from $\langle U \setminus \{i\} \rangle = \{o, a, b, ab, ba\}$.
Notice also that $\{a\}\ and\ \{b\}$ are excluded from the subsemigroup $\langle U \setminus \{a\} \rangle = \{o, i, b\}$ and $\langle U \setminus \{b\} \rangle = \{o, i, a\}$ respectively. We conclude that the subset U is independent.

Next, we show that U is a maximal independent subset of the semigroup S.
Notice that $U = \{i, a, b\}$ becomes dependent if we include any of $o\ or\ ab\ or\ ba$in U:
The subsemigroup generated by $\{o, i, a, b\} \setminus \{o\}$ is $\{o, i, a, b, ab, ba\} \supseteq \{o\}$
The subsemigroup generated by $\{ab, i, a, b\} \setminus \{ab\}$ is $\{o, i, a, b, ab, ba\} \supseteq \{ab\}$
The subsemigroup generated by $\{ba, i, a, b\} \setminus \{ba\}$ is $\{o, i, a, b, ab, ba\} \supseteq \{ba\}$
That is;
For $\{o, i, a, b\}$: $\{o\}$ is in the subsemigroup generated by $\{o, i, a, b\} \setminus \{o\}$
For $\{ab, i, a, b\}$: $\{ab\}$ is in the subsemigroup generated by $\{ab, i, a, b\} \setminus \{ab\}$
For $\{ba, i, a, b\}$: $\{ba\}$ is in the subsemigroup generated by $\{ba, i, a, b\} \setminus \{ba\}$
Therefore U is a maximal independent subset of S. There may exist other Maximal Independent subsets in S but the number of elements in each must be the same. The rank of S is 3 - the number of elements in U. Each maximal independent subset is called a basis of $\langle U\rangle = \{o, i, a, b, ab, ba\} = S$.

II. DEFINITION (Rank of Semigroup by Howie and Ribeiro (6))

Given a finite semigroup$S$,
(1)
$r_1(S) = max\{k :$
$every\ subset\ U\ of\ cardinality\ k\ in\ S\ is\ independent\}$
.
(2) $r_2(S) = min\{|U| : U \subseteq S, \langle U\rangle = S\}$.
(3)
$r_3(S) = max\{|U| : U \subseteq S, \langle U\rangle =$
$S, U\ is\ independent\}$.
(4) $r_4(S) = max\{|U| : U \subseteq S, U\ is\ independent\}$.
(5)
$r_5(S) = min\{k :$
$every\ subset\ U\ of\ cardinality\ k\ in\ S\ generates\ S\}$
.
For a finite semigroup$S$, it can be observed that

$$r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$$

Thus, $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$are, respectively, known as small rank, lower rank, intermediate rank, upper rank and large rank of S. Here, the lower Rank is what is usually called the rank which has been extensively studied.

2.1. **Remark**
Our approach is to show first with construction of examples for semigroups - finite and infinite, commutative and non-commutative, where there exists finite basis and the number of independent elements is larger than the basis. A Basis is an independent subset and also a maximal generating set. The idea we intend to communicate here is that
i. As already known with semigroups, the minimal generating set is **not** always equal to the maximal independent subset
ii. Furthermore, the maximal independent subset does not always exist and thus fails to be a worthy tool for characterising ranks.

2.2. **Definition**
Let $\{S,*\}$be a Semigroup. Let $H \subset S$be a set. The closure of Hin $S$is the Semigroup

$$Cl(H) := \left\{ \prod_{j=1}^{k} a_j \;\middle|\; \forall \{a_j\}_{j=1}^{k} \subset H, 1 \leq k \leq \infty \right\}$$

2.3. **Definition**
A subset $G \subset S$is a generator system in S if
$Cl(G) = S$.

2.4. **Definition**
A subset $H \subset S$is an independent system of semi-ring elements in Sif

$$\forall x \in H, x \notin Cl(H \setminus \{x\}).$$

2.5. **Definition**
A subset $B \subset S$is a basis in Sif
1. $B$is a generator system in S;
2. $B$is an independent system.

2.0. **Infinite Semigroup examples.**
In this section we will construct infinite semigroups generated by two or finite bases which
have infinite system of independent systems.

2.1. **Example for commutative semigroup.**
Let a commutative semigroup be generated by a basis $B := \{a, b\}$ of two independent elements.

Assume that both $a, b \in B$generate infinite cyclic Semigroups: $C_a := \{a^k | 1 \leq k < \infty\}$
and$C_b := \{b^k | 1 \leq k < \infty\}$.
Let the Semigroup$S_c := \{a^n b^m | 1 \leq n + m < \infty, n, m \in N\}$with the multiplication $(a^n b^m) * (a^p b^q) := a^{n+p} b^{m+q}$, $\forall 1 \leq (n + m)(p + q)\ and\ \{n, m, p, q\} \subset N$.

2.2. **Theorem**
$H := \{ab^j | 1 \leq j < \infty\} \subset S_c$is an infinite independent system in S_c.
Proof. Let $ab^j \in H$for $j \in N\setminus\{0\}$.
Then $ab^j \in H\setminus\{ab^j\}$and $\forall x \in Cl(H\setminus\{ab^j\})ab^j \neq x \in H$
Otherwise $x \notin H \Longrightarrow x = a^p b^q$, $p \geq 2$. Hence $ab^j \notin Cl(H \setminus \{ab^j\}) \forall 1 \leq j < \infty$.

2.3. **Conclusion**
There exists a commutative semigroup generated with a two element basis which has an infinite system of independent subset.

2.4. **Example of non-commutative semigroup.**
Let a non-commutative semigroup be generated by a basis $B := \{a, b\}$of two independent

elements as a free semigroup of finite words created from B:

$S_F := \{a_1 a_2 \ldots a_k \mid a_j \in B, 1 \le j \le k, 1 \le k < 1\}$.

As in a free semigroup/group the operation is concatenation of words as elements of the semigroupS_F. Both $a\ and\ b$generates cyclic subsemigroupsin S_F,

$S_{F,a} := \{a^j, 1 \le j < \infty\}\ and\ S_{F,b} := \{b^j, 1 \le j < \infty\}$.

2.5. **Theorem**

$H := \{ab^j \mid 1 \le j < \infty\} \subset S_F$is an infinite independent system in S_F.

Proof.

Let $ab^j \in H$ for $a\ j \in N\backslash\{0\}$. Then $ab^j \notin H \setminus \{ab^j\}$and $\forall x \in Cl(H \setminus \{ab^j\}) \Longrightarrow ab^j \ne x \in H$.

Otherwise $x \notin H \Longrightarrow x = \prod_{t=1}^{s} ab^{q_t}, s \ge 2, q_t \ge 1, 1 \le t \le s$.

Hence

$ab^j \notin Cl(H \setminus \{ab^j\})\ \forall 1 \le j < \infty$.

2.6. **Conclusion.**

There exists a non-commutative semigroup generated with a two element basis which has an infinite system of independent subset.

III. THE CASE OF FINITE SEMIGROUPS: INTRODUCTION

First we will consider some methods to construct finite semigroups which we will utilize in section 4 to give examples of cases involving finite semigroups.

3.1. **Semigroup homomorphism.**

Let $\{S_1, *_1\}$ and $\{S_2, *_2\}$ be two semigroups.

3.2. **Definition.**

A mapping $h: S_1 \to S_2$is a semigroup homomorphism if

$\forall a, b \in S_1, \square(a *_1 b) = h(a) *_2 h(b)$.

3.3. **Lemma.**

If $h: S_1 \to S_2$is a semi-ring homomorphism, then the range $h(S_1) =: R(h) \subset S_2$is a semi-ring. If $R(h) = S_2$then h is onto.

3.4. **Remark**.

We may assume that $h: S_1 \to S_2$is onto hence the range is S_2.

3.5. **Definition**

Let $h: S_1 \to S_2$be a semigroup homomorphism. Then $h^*(u) := h^{-1}(u) =: u^* \subset S_1,\ \forall u \in S_2$.

3.6. **Theorem**

i. $S_1 \supset u^* = \square^{-1}(u) \ne \emptyset, \forall u \in S_2$;

ii. $Let\ S_2^* := \{u^* | \square^{-1}(u) =: u^*, \forall u \in S_2\}$. Then $\forall u \in S_2, \square^*(u) : u \in S_2 \to u^* \in P(S_1)$ is a bijective mapping from $S_2 \to P(S_1)$. Moreover, $u^* \cap v^* = \emptyset, \forall u, v \in S_2$, u≠ v.

iii. $\bigcup_{u \in S_2} u^* = S_1$

iv. Let $x \in u^*, y \in v^*, u, v \in S_2$. Then $f(x) = u \in S_2,\ f(y) = v \in S_2$and $f(x *_1 y) = f(x) *_2 f(y) = u *_2 v$. Hence $x *_1 y \in \big(f(x) *_2 f(y)\big)^* = (u *_2 v)^*$.

Proof.

i. Since h is a mapping, the pre-image exists and is a Boolean homomorphism in the Boolean algebra of the power set $P(S_1)$. Assuming that $h$is onto, $\exists x_u \in S_1$such that

$f(x_u) = u, \forall u \in S_2 hence\ x_u \in \square^{-1}(u) \ne \emptyset$;

ii. By $\square^{-1}$being a Boolean homomorphism, $\forall u, v \in S_2,\ u \ne v$follows that

$\{u\} \cap \{v\} = \emptyset \Longrightarrow \emptyset = h^{-1}(\{u\} \cap \{v\})$

$= h^{-1}(\{u\}) \cap h^{-1}(\{v\}) = u^* \cap v^*$

iii. $\forall x \in S_1,\ h(x) = u \in R(h)$. Hence $x \in h^{-1}(u)$. Hence $\bigcup_{u \in R(h) \subset S_2} u^* = R(h) \subset S_1$

iv. This point actually follows from the way the statement is formulated.

3.7. **Definition**

Let $h: S_1 \to S_2$be a semigroup homomorphism. Then $\forall x, y \in S_1,\ x \sim_h y \Leftrightarrow h(x) = h(y) \in S_2$.

3.8. **Definition** (Equivalence Relation)

Let S be a semigroup. An equivalence relation $\sim$on S is a subset of $S \times S$ of ordered pair of elements of S that satisfy the following properties:

i. Reflexive: $a \sim a$ for all $a \in S$,

ii. Symmetric: $a \sim b$ implies $b \sim a$ for all $a, b \in S$

iii. Transitive: $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in S$

3.9. **Lemma**

The relation $\sim_h \subset S_1 \times S_1$is an equivalence relation and the partition by equivalence classes is $\hat{x} = \square(x)^*. \forall s \in S_1$

Proof.

The proof of $\sim_h \subset S_1 \times S_1$being equivalence relation is left to the reader, follows from the definition.

Let $x \in S_1$be selected. Then $\hat{x} := \{y \mid h(y) = h(x) =: u_x \in R(h)\} = \{y \mid h(y) = u_x\} = h^{-1}(u_x) = (u_x)^*, u_x \in R(h) \subset S_2$.

Conversely, if $u \in R(h) \subset S_2$then $u^* := h^{-1}(u) := \{y \mid h(y) = u\}$. Hence if$x, y \in u^* \Longrightarrow h(x) = h(y) = u \Longrightarrow x \sim_h y$,hence $u^* = \hat{x},\ \forall x \in u^*$. This completes the proof.

3.10. **Operation on the equivalence classes.**

Let us denote the set of equivalence classes by $S_2/\sim_h =: S_2^*$.

3.11. **Definition**
Let $u^*, v^* \in S_2$. Then the product of $u^* and\ v^*$is defined by
$u^* * v^* := (h(x) *_2 h(y))^*, x \in u^*, y \in v^*.$

3.12. **Lemma**
The product is well defined, $S_2^*$with the operation is a semigroup and the
semigroup is isomorphic to $R(h) \subset S_2$

Proof.
First, we show that the product is well defined:
Let $u^*, v^* \in S_2^*$. Then the product of $u^* and\ v^*$is defined by

$$u^* * v^* := \left(h(x) *_2 h(y)\right)^*, x \in u^*, y \in v^*.$$

We have to show that if $\bar{x} \in u^*$and $\bar{y} \in v^*$are selected then $(h(x) *_2 h(y))^* = (h(\bar{x}) *_2 h(\bar{y}))^*$. By $x, \bar{x} \in u^* \Longrightarrow h(x) = h(\bar{x}) = u$
and$y, \bar{y} \in v^* \Longrightarrow h(y) = h(\bar{y}) = v$. Hence
$h(x) *_2 h(y) = u *_2 v = h(\bar{x}) *_2 h(\bar{y}) \Longrightarrow u^* * v^* = (u *_2 v)^*$.

Next we show that it is a Semigroup:
Let $u, v, w \in S_2$. Then $u^* * (v^* * w^*) = u^* * (v *_2 w)^* = \left(u *_2 (v *_2 w)\right)^*$
$= \left((u *_2 v) *_2 w\right)^* = (u *_2 v)^* * w^* =$
$(u^* *_2 v^*) * w^*$proves the associativity.

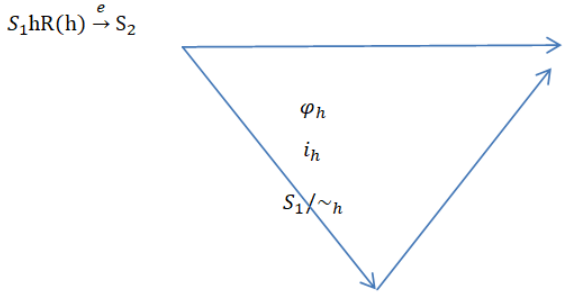
Finally, we show that S_2^* is isomorphic to $R(h) \subset S_2$: From statements (ii) and (iii) section 3.6, it follows that the mapping $u \in R(h) \subset S_2 \to u^* \subset P(S_1)$ is one to one and the union of the images is S_2. In points one and two of this proof we saw that $\forall u, v \in R(h),\ u^* * v^* = (u *_2 v)$that proves the isomorphism

IV. INTRODUCTION

In the previous sections we prepared the formulation of the so-called isomorphism theorem. In this section, we state the isomorphism theorem as well as use the idea to construct an example for finite commutative semigroup.

Let S_1, S_2be semi-rings, let $h: S_1 \to R(h) \subset S_2$be a homomorphism. Let $i_h: u^* \in S_2^* \to R(h)$
defined by the isomorphism $u^* \to u \in R(h)$. Let $\varphi_h: S_1 \to S_1/\sim_h$the factor mapping for the
equivalence relation $\sim_h$. Then the isomorphism theorem is stated as follows:

4.1. **Isomorphism theorem**
Let S_1, S_2be semi-rings, let □ : $S_1 \to R(□) \subset S_2$be a homomorphism. Then the following diagram is commutative:



Proof. The proof is left to the reader, it is well prepared in the earlier sections.

4.2. **Finite commutative group examples.**
Let $\{S,*\}$be an infinite commutative semigroup generated by one element, which means it is
an infinite cyclic semigroup. Let

$$\varphi_{p,q}(n) := \begin{cases} n, & n < p \\ \left((n-p) mod\ q\right) + p, & n \geq p \end{cases} \quad \forall n \in N$$

4.3. **Lemma**
Let $a \in S$ be the generator element. Then the formal set

$$S_{p,q} := \{b^{\varphi_{p,q}(n)} | \ n \in N, \quad n > 0\}$$

is a semigroup for the multiplication $b^{\varphi_{p,q}(n)} * b^{\varphi_{p,q}(m)} = b^{\varphi_{p,q}(n+m)}$.
What is more, $h: a^n \in S \to b^{\varphi_{p,q}(n)}$is a semigrouphomomorphism.
Proof.
The only critical issue is the properties of $\varphi_{p,q}(n)$namely when $n = p + q$then
$\varphi_{p,q}(p+q) = \varphi_{p,q}(p)$.
Hence $S_{p.q}$is a finite semigroup which is the homomorphic image of an infinite cyclic
semigroup.

4.4. **Theorem**
Let $S_c$be the infinite semigroup as defined in 2.9 with independent elements $a, b \in S$.
Let $H := \{ab^j | \ 1 \leq j < \infty\} \subset S_c$be an infinite independent system in S_c. Let
$\bar{a}, \bar{b}$symbols and let a commutative semigroup be
$S_{\bar{a},\bar{b},p,q} := \{\bar{a}^{\varphi_{p,q}(n)} \bar{b}^{\varphi_{p,q}(m)} | \ n, m \in N, n + m \geq 1\}$.
Then $\bar{H} := \{\bar{a}\bar{b}^{\varphi_{p,q}(n)}\}$is an independent system in $S_{\bar{a},\bar{b},p,q}$. The finite semigroup is a homomorphic image of the $S_c$semigroup.

CONCLUSION

In summary, existing works began with attempts at characterizing rank using previous characterization as well as examples of ranks of simpler algebraic structures as guide, resulted in the amended characterisations by Howie and Ribeiro. We defer in this approach and believe that the appropriate rank notions need not refer to cardinality, rather than number in general. The reason is as follows:

Algebra has its methodology and semigroups are not exception. We look at things of this kind from a structural approach. We believe that all the confusion about huge families of ranks should not have ever come up if generator system had been defined as smallest sub-semigroups containing the generator set. This explains the differences in a natural way and no need to talk about vector spaces etc. The major difference between "old" style and modern algebraic approach is that first we look at structures and then examples and not first forest of examples and the forest cannot be seen because of the trees, which is the very limiting approach that led to the current characterization of rank. We intend to avoid that limitation by looking at a structural approach without primarily relying on examples for guide.

Also, we consider something extended because ranks/dimension/generator sets are in the first 20 pages at most in chapters of an algebraic field. Cardinal/ordinal numbers and numbers are simply numbers, first introduced by Cantor over 130 years ago. Number means what it ought to be and we do not even question which concept is to be used. So the challenges of journeying into a hitch through a less fancy approach should not give us the liberty to redefine numbers (or restrict ourselves to a particular type of numbers) in order to be correct as this will not allow us arrive at a more general answer to the question of characterization for rank. The notion of rank should rather be dictated by the nature of the problem and not the nature of the problem dictating the notion of rank.

Axiomatic set theory puts its vote on ordinal numbers by which it lists all sets with an "index' so that the concept of set is fixed and antinomies die. We however have one important observation: independent sets and generator sets are different concepts which may come together in structures where there is a rich toolkit for the formation of elements from a set of elements.

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