A STOCHASTIC PDE UNDER FRACTIONAL BROWNIAN MOTION

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Abstract- This paper studies a class of stochastic PDEs driven by fractional Brownian mo- tion with Hurst index H 2 (1 2 ; 1) in Hilbert space. We prove the existence theorem viaxed point theorem and theory of analytic resolvent operators.

Index terms- Fractional Brownian mo-tion, xed point theory, stochastic integro-dierential equations, existence theorem.

I. BACKGROUND

Fractional Brownian motion is a family of centered Gaussian processes with continuous
sample paths indexed by the Hurst parameter H 2 (0; 1). In modern probability and statistical modeling, fractional Brownian motion is an important tool (see [1]
and [10] for references). It has also created many active research areas in applied
physics, chemistry, biology, medicine, economics, etc. ([4, 5, 6]). Meanwhile, people
have been studying neutral stochastic partial di_erential equations and integrodi _erential equations with resolvent operators. For example, [2, 7, 8, 9] obtained existence, global attracting set and stability of solutions for neutral stochastic PDE in in-_nite dimensional spaces. We are aware of few results on stochastic PDEs and integrodi _erential equations with delay driven by fractional Brownian motion. For example, Caraballo el al. [2] studied existence and exponential behavior of solutions to stochastic delay evolution equations with an fractional Brownian motion. Ferrante and
Rovira [3] established existence and uniqueness of solutions to delayed stochastic differential equations under fractional Brownian motion for H > 1 2 , etc. In this paper,
we prove existence theorem for mild solutions in the following class of neutral stochastic integro-di_erential equations under fractional Brownian motion in Hilbert space (f; g : J _ H ! H, h : J ! L0 2 are appropriate mappings to be speci_ed), see Fig. 1, where A : D(A) _ H ! H is the in_nitesimal generator of an resolvent family fR(t)gt_0 and the state x(_)
takes values in a separable real Hilbert space H. Here BH is a fractional Brownian motion in a separable real Hilbert space K; r; _ : J ! [0; _] are continuous functions
and K(t) : D(K(t)) _ H ! H is a bounded linear operator.

$$d[x(t)+f(t,x(t-r(t)))]=A[x(t)+\int_0^t K(t-s)x(s)ds]dt + g(t,x(t-\rho(t)))dt+h(t)dB^H(t)$$
$$x_0(t)=\varphi(t)\in\mathbb{C}_0^{\mathcal{F}_0}([-\tau,0],\mathcal{L}^2(\Omega,\mathbb{H})),\quad -\tau\le t\le 0,\quad \tau>0,a.s.$$

Fig. 1.

Let $(\mathbb{H},\|\cdot\|_{\mathbb{H}},\langle\cdot,\cdot\rangle)$ and $(\mathbb{K},\|\cdot\|_{\mathbb{K}},\langle\cdot,\cdot\rangle)$ be two real separable Hilbert spaces, with their vector norms and inner products respectively. Let $\mathcal{L}(\mathbb{K};\mathbb{H})$ be the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$ under the usual operator norm $\|\cdot\|$ and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in J},\mathbf{P})$ be a complete filtered probability space continuous from the right. Finally, $\{B^H(t)\}_{t\in J}$ is a fractional Brownian motion to the filtration $\{\mathcal{F}_t\}_{t\in J}$.

We note that an one-dimensional fractional Brownian motion with Hurst parameter $H\in(0,1)$ is a centered Gaussian process $\beta^H=\{\beta^H(t)\}_{t\in J}$, $\beta^{\frac{1}{2}}$ is a standard Brownian motion and $\beta^H(t)$ with $H\in(\frac{1}{2},1)$ has the Volterra representation $\beta^H(t)=\int_0^t K_H(t,s)d\beta(s)$ where $\beta=\{\beta(t)\}_{t\in J}$ is a Wiener process. The fractional Wiener integral of $\Psi: J\to\mathcal{L}_2^0$ is defined by $\int_0^t\Psi(s)dB^H(s)=\sum_{n=1}^\infty\int_0^t\sqrt{\lambda_n}K_H^*(\Psi e_n)d\beta_n(s)$ where β_n is the standard Brownian motion used to present β_n^H. It was shown that the above definition is valid as a $\mathbb{H}$-valued random variable provided that $\int_J\|\Psi(s)\|_{\mathcal{L}_2^0}^2ds<\infty$. In this case, we have $\mathbf{E}\left\|\Psi(s)dB^H(s)\right\|^2\le 2Ht^{2H-1}\int_0^t\|\Psi(s)\|_{\mathcal{L}_2^0}^2ds$.

Let X,Z be two Banach spaces such that $\|z\|_Z:=\|Az\|_X+\|z\|_X$ for all $z\in Z$; A and $K(t)$ be closed linear operators on X where

$A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup on X; $K(t) : D(K(t)) \subseteq X \to X$ is a closed linear operator and the map $t \to K(t)z$ is bounded with a continuous and bounded derivative. By [11], the following integro-differential abstract Cauchy problem $\frac{dx(t)}{dt} = A\Big[x(t) + \int_0^t K(t-s)x(s)ds\Big], x(0) = x_0 \in X$ has an associated resolvent operator of bounded linear operators $(R(t))_{t\in J}$ on X.

An $\mathcal{F}_t$-adapted stochastic process $x : [-\tau, T] \to \mathbb{H}$, $0 < T < +\infty$ is called a mild solution of (1.1) on $[-\tau, T]$ if $x_0(\cdot) = \varphi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$ on $[-\tau, 0]$ a.s., and for each $s \in [0, T)$ the function $AR(t-s)f(s, x(s-r(s)))$ is integrable such that $x(\cdot) \in \mathcal{C}([-\tau, T], \mathcal{L}^2(\Omega, \mathbb{H}))$ and $x(t)$ satisfies the integral equation in Fig. 2.

II. EXISTENCE THEOREM

Theorem 2.1. Under suitable conditions, there exists a unique mild solution to (1.1) on [; T] provided that

$$4\Big[M_f^2\Big(\|A^{-\alpha}\|^2 + \frac{T^{2\alpha+1}M_{1-\alpha}^2}{2\alpha-1}[1+TM_k^2]\Big) + T^2M_g^2C^2\Big] < 1. \quad (2.1)$$

Proof. We consider the space $\Upsilon_T = \{x \in \mathbb{C}_T : x(s) = \varphi(s), s \in [-\tau, 0]\}$. Υ_T is a closed subset of $\mathbb{C}_T$ endowed with the norm $\|\cdot\|_{\mathbb{C}_T}$ and define the operator $\Pi : \Upsilon_T \to \Upsilon_T$

$$x(t) = R(t)[\varphi(0) + f(0, \varphi(-r(0)))] - \int_0^t AR(t-s)f(s, x(s-r(s)))ds - f(t, x(t-r(t))) - \int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi, x(\xi-r(\xi)))d\xi ds + \int_0^t R(t-s)g(s, x(s-\rho(s)))ds + \int_0^t R(t-s)h(s)dB^H(s), \quad \mathbf{P}-a.s..$$

Fig. 2.

$$(\Pi x)(t) = R(t)[\varphi(0) + f(0, \varphi(-r(0)))] - f(t, x(t-r(t))) - \int_0^t AR(t-s)f(s, x(s-r(s)))ds - \int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi, x(\xi-r(\xi)))d\xi ds + \int_0^t R(t-s)g(s, x(s-\rho(s)))ds + \int_0^t R(t-s)h(s)dB^H(s), \quad t \in J,$$

Fig. 3.

by $(\Pi x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and we have the formula in Fig. 3.

It suffices to show that the operator Π has a unique fixed point in Υ_T. Clearly Π_2 is continuous on J in the $\mathcal{L}^2(\Omega, \mathbb{H})$-sense. Let $t \in (0, T)$ and $|\epsilon|$ be sufficiently small. Then for any fixed $x \in \mathbb{C}_T$, we have Fig. 4.

Strong continuity of $R(t)$ implies Fig. 5 Based on Lebesgue dominated theorem, we have $\lim_{\epsilon\to 0}\mathbf{E}\|Q_1(\epsilon)\|^2 = 0$ and

$$\mathbf{E}\|Q_2(\epsilon)\|^2 \leq \|A^{-\alpha}\|^2\mathbf{E}\|A^\alpha f(t+\epsilon, x(t+\epsilon-r(t+\epsilon))) - A^\alpha f(t, x(t-r(t)))\|^2.$$

Since the operator $A^{-\alpha}$ is bounded, we conclude that $\lim_{\epsilon\to 0}\mathbf{E}\|Q_2(\epsilon)\|^2 = 0$.

Without loss of generality, assume $\epsilon > 0$ (the case $\epsilon < 0$ is quite similar). We have Fig. 6

Hölder inequality on $Q_{31}(\epsilon)$ and Lebesgue dominated theorem imply that $\lim_{\epsilon\to 0}\mathbf{E}\|Q_{31}(\epsilon)\|^2 = 0$ and

$$\mathbf{E}\|Q_{32}(\epsilon)\|^2 \leq \frac{L_f M_{1-\alpha}^2}{2\alpha-1}\epsilon^{2\alpha}\int_0^T(\mathbf{E}\|x(s-r(s))\|^2+1)ds \xrightarrow{\epsilon\to 0} 0.$$

thus $\lim_{\epsilon\to 0}\mathbf{E}\|Q_3(\epsilon)\|^2 = 0$.

For $Q_4(\epsilon)$ we assume that $\epsilon > 0$, and we have Fig. 7. By Hölder inequality, we get Fig. 8

Again Lebesgue dominated theorem leads to $\lim_{\epsilon\to 0}\mathbf{E}\|Q_{41}(\epsilon)\|^2 = 0$. Similarly,

$$\mathbf{E}\|Q_{42}(\epsilon)\|^2 \leq \frac{tL_f M_k^2 M_{1-\alpha}^2}{2\alpha-1}\epsilon^{2\alpha}\int_0^T (\mathbf{E}\|x(s-r(s))\|^2+1)ds \xrightarrow{\epsilon\to 0} 0.$$

Therefore, $\lim_{\epsilon\to 0}\mathbf{E}\|Q_4(\epsilon)\|^2 = 0$. The computation of $Q_3(\epsilon)$ also implies that $\lim_{\epsilon\to 0}\mathbf{E}\|Q_5(\epsilon)\|^2 = 0$.

$$\begin{aligned}
\|(\Pi x)(t+\epsilon)-(\Pi x)(t)\| &\leq \|R(t+\epsilon)-R(t)[\varphi(0)+f(0,\varphi(-r(0)))]\| \\
&+\|f(t+\epsilon,x(t+\epsilon-r(t+\epsilon)))-f(t,x(t-r(t)))\| \\
+\Big\|\int_0^{t+\epsilon} AR(t+\epsilon-s)f(s,x(s-r(s)))ds &- \int_0^t AR(t-s)f(s,x(s-r(s)))ds\Big\| \\
+\Big\|\int_0^{t+\epsilon} AR(t+\epsilon-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds & \\
-\int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds\Big\| & \\
+\Big\|\int_0^{t+\epsilon} R(t+\epsilon-s)g(s,x(s-\rho(s)))ds - \int_0^t R(t-s)g(s,x(s-\rho(s)))ds\Big\| & \\
+\Big\|\int_0^{t+\epsilon} R(t+\epsilon-s)h(s)dB^H(s) - \int_0^t R(t-s)h(s)dB^H(s)\Big\| &= \sum_{k=1}^6 Q_k(t)
\end{aligned}$$

Fig. 4.

$$\lim_{\epsilon\to 0}[R(t+\epsilon)-R(t)][\varphi(0)+f(0,\varphi(-r(0)))]=0,$$
$$\|[R(t+\epsilon)-R(t)][\varphi(0)+f(0,\varphi(-r(0)))]\| \leq 2C\|\varphi(0)+f(0,\varphi(-r(0)))\| \in \mathcal{L}^2(\Omega).$$

Fig. 5.

$$\begin{aligned}
Q_3(\epsilon) &\leq \Big\|\int_0^t [A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]A^\alpha f(s,x(s-r(s)))ds\Big\| \\
&+\Big\|\int_t^{t+\epsilon}[A^{1-\alpha}R(t+\epsilon-s)A^\alpha f(s,x(s-r(s)))ds\Big\| = Q_{31}(\epsilon)+Q_{32}(\epsilon).
\end{aligned}$$

Fig. 6.

$$\begin{aligned}
Q_4(\epsilon) &\leq \Big\|\int_0^t [A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi ds\Big\| \\
&+\Big\|\int_t^{t+\epsilon}[A^{1-\alpha}R(t+\epsilon-s)\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi ds\Big\| \\
&= Q_{41}(\epsilon)+Q_{42}(\epsilon).
\end{aligned}$$

Fig. 7.

To take care of Q_6, let

$$Q_6(\epsilon) \leq \Big\|\int_0^t [R(t+\epsilon-s)-R(t-s)]h(s)dB^H(s)\Big\|$$

$$\mathbf{E}\|Q_{41}(\epsilon)\|^2$$
$$\leq t\mathbf{E}\int_0^t \|[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^{\alpha}f(s,x(s-r(s)))d\xi\|^2 ds.$$

Since $\lim_{\epsilon\to 0}[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^{\alpha}f(s,x(s-r(s)))d\xi = 0$, we have

$$\|[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^{\alpha}f(s,x(s-r(s)))d\xi\|$$
$$\leq tM_kM_{1-\alpha}\big((t+\epsilon-s)^{\alpha-1}+(t-s)^{\alpha-1}\big)\|A^{\alpha}f(s,x(s-r(s)))\| \in \mathcal{L}^2([0,t]\times\Omega).$$

Fig. 8.

$$+\Big\|\int_t^{t+\epsilon} R(t+\epsilon-s)h(s)dB^H(s)\Big\|$$
$$= Q_{61}(\epsilon)+Q_{62}(\epsilon),$$

and

$$\mathbf{E}\|Q_{61}(\epsilon)\|^2 \leq$$
$$2Ht^{2H-1}\int_0^t \|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0}d(s).$$

Since $R(t)$ is strongly continuous, the following limit and inequality hold for each $s\in[0,t]$.

$$\lim_{\epsilon\to 0}\|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0}=0$$
$$\|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0}\leq$$
$$2C^2\|h(s)\|^2_{\mathcal{L}_2^0}\in\mathcal{L}^1(J,ds).$$

Therefore, we have $\lim_{\epsilon\to 0}\mathbf{E}\|Q_{61}(\epsilon)\|^2=0$.

Similarly,

$$\mathbf{E}\|Q_{62}(\epsilon)\|^2 \leq$$
$$2HC^2\epsilon^{2H-1}\int_t^{t+\epsilon}\|h(s)\|^2_{\mathcal{L}_2^0}d(s)\xrightarrow{\epsilon\to 0}0.$$

and we are able to conclude that

$$\lim_{\epsilon\to 0}\mathbf{E}\big\|(\Pi x)(t+\epsilon)-(\Pi x)(t)\big\|^2=0.$$

Here we notice that the function $t\to(\Pi x)(t)$ is continuous on J in the $\mathcal{L}^2$-sense.

Since Π_2 is a contraction mapping in $\Upsilon_{\overline{T}}$ with some small enough $\overline{T}<T$, let $x,y\in\Upsilon_T$. By Hölder's inequality, for all $t\in J$ we have Fig. 9. Taking supremum over t we obtain Fig. 10 This implies that Π is a contractive mapping on $\Upsilon_{\overline{T}}$. By Banach fixed point principle we conclude that there exists a unique fixed point, which is a mild solution of (1.1) on $[-\tau,\overline{T}]$. This procedure can be repeated to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. Therefore, the proof of Theorem 2.1 is completed.

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$$\begin{aligned}
\mathbf{E}\big\|(\Pi x)(t)-(\Pi y)(t)\big\|^2 &\le 4\mathbf{E}\big\|f(t,x(t-r(t)))-f(t,y(t-r(t)))\big\|^2 \\
&\quad+4\mathbf{E}\Big\|\int_0^t AR(t-s)[f(s,x(s-r(s)))-f(s,y(s-r(s)))]ds\Big\|^2 \\
&\quad+4\mathbf{E}\Big\|\int_0^t AR(t-s)\int_0^s K(s-\xi)[f(\xi,x(\xi-r(\xi)))-f(\xi,y(\xi-r(\xi)))]d\xi ds\Big\|^2 \\
&\quad+4\mathbf{E}\Big\|\int_0^t R(t-s)[g(s,x(s-\rho(s)))-g(s,y(s-\rho(s)))]ds\Big\|^2 \\
&\le 4\|A^{-\alpha}\|^2M_f^2\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2+\frac{4T^{2\alpha}M_f^2M_{1-\alpha}^2}{2\alpha-1}\int_0^t\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2ds \\
&\quad+\frac{4T^{2\alpha+1}M_f^2M_k^2M_{1-\alpha}^2}{2\alpha-1}\int_0^t\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2ds \\
&\quad+4TM_g^2C^2\int_0^t\mathbf{E}\|x(s-\rho(s))-y(s-\rho(s))\|^2ds.
\end{aligned}$$

Fig. 9.

$$\|(\Pi x)-(\Pi y)\|_{\mathbb{C}_T}^2\le 4\Big[M_f^2\Big(\|A^{-\alpha}\|^2+\frac{T^{2\alpha+1}M_{1-\alpha}^2}{2\alpha-1}[1+TM_k^2]\Big)+T^2M_g^2C^2\Big]\|x-y\|_{\mathbb{C}_T}^2.$$

Fig. 10.

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