FRACTIONAL CALCULUS IN SOLVING ENGINEERING PROBLEMS

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Abstract - Fractional calculus is a rapidly growing field both in theory and applications in the real world problems to explain several physical phenomena. Fractional order system or systems containing fractional derivatives and integration which has a non- local character. The present paper discusses several definitions of fractional derivatives including a discussion of notation, operators in the sense of Riemann-Liouville, Caputo, Grunwald- Letnikov etc. The concept of fractional order differential equation is also introduced. A specific problem on ordinary fractional order differential equation related to RC circuit with semi-infinite lossy transmission line is studied. A very new approach Adomain Decomposition Method (ADM) is utilized to solve the problem.

Index Terms - Fractional Differential Equation; Generalized dynamic system; Adomain Decomposition method.

I. INTRODUCTION

The idea of fractional calculus (or often called non-integer calculus) was first introduced in the year 1695 by L'Hospitals. In a letter dated 30th September, 1695, L'Hospitals wrote to Leibnit'z asking him about the derivative of $\frac{d^n y}{dx^n}$ when n=1/2 and for y=x. Leibnitz replied that $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$ and responded : "this is an apparent paradox from which one day useful consequence will be drawn". After that mathematicians like Fourier, Euler, Laplace have studied fractional calculus (integration and differentiation) and its mathematical consequences [12, 13, 14]. The order of the fractional derivative α where $0<\alpha<1$ classifies and quantifies the influence of past history. Note that fractional derivative definitions have inbuilt integration, meaning that the operation is non-local, requires information about past states. For unity value of the fractional order (or integer of order one) the influence of past history is minimal, in the sense that, then the process evolution enters only through the present state. Small value of α corresponds to a strong influence of past history. Many authors have developed and defined the concept of fractional integrals and derivatives (often called differintegrals) in their own way. Of them, the name of Riemann- Liouville, Weyl and Grunwald-Letnikov are worth mentioning. However, Caputo reformulated the more "classic" definition of the Riemann-Liouville fractional derivative in order to use integer order initial condition to solve fractional order differential equation [15]. It is in 1996, Kolwankar and Gangal reformulated again, the Riemann-Liouville fractional derivative in order to differentiate nowhere differentiable fractal functions [6]. Several applications and physical manifestations of fractional calculus related to science and technology have been found in 20th century. The applications like, Abel's fractional integral equation of Tatochrome, Modelling of speech signals [1], Modelling of cardiac tissue electrode interface [7, 8], Fractional differentiation of edge detection related to image processing [10], the solution of time-dependent viscous- diffusion fluid problems [5] are worth mentioning. Some other applications are sound waves propagation in rigid porous materials [4], lateral and longitudinal control of autonomous vessels [19, 20], ultrasonic wave propagation in human cancellous bone [17], applications in viscous elasticity [9, 18], polymer science [2], fractal phenomena [16].

Several analytical methods are used to solve fractional order differential equation. The Laplace transform method is a very popular technique although other analytical techniques do exist. Some more analytical methods are Mellin's transform, power series expansion, method approach using Green's function, Babenko's symbolic method, complementary orthogonal polynomial method, Reisz fraction potential method etc. In this connection several functions of fractional calculus viz., Beta functions, Gamma functions, Mittag-Leffer function, Miller Ross function, Agarwal function etc. is addressed [3, 11, 12, 13, 15]. However, a very new approach to solve fractional differential equation is Adomain Decomposition Method (ADM). In this paper, we solve a fractional differential equation of mass spring system using ADM. As an application of this a RC circuit with semi-infinite lossy transmission line is studied.

The whole matter of the paper is organized as follows:

Section I is the introductory one. Section II gives the definitions and results of fractional integrations and derivatives in terms of several authors. In section III a generalized dynamic system and the idea of solution of fractional differential equation using ADM is discussed. Section IV gives an example problem of third order mass spring system with half order element using ADM. Finally in section V, a discussion is made for a third order RC circuit with

semi-infinite lossy cable acts as half order element connected to a CRO probe.

II. SOME KNOWN DEFINITIONS OF FRACTIONAL INTERGRATIONS AND DIFFERENTIATIONS [3,15]

A. Fractional Integration:

A repeated n-folded integration is defined as

$$D^{-n} f(t) = I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-u)^{n-1} f(u)du$$

For any real number α>0, the repeated α- fold integration is defined as

$$D^{-\alpha} f(t) = I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u)du$$

The following are some definitions of fractional integration in the sense of several authors.

(a) Riemann-Liouville (RL) Integration: For α>0

(i) Forward Integration:

$${}_a I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1} f(u)du$$

(ii) Backward Integration:

$${}_x I_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (u-x)^{\alpha-1} f(u)du$$

(b) Weyl's (W) Integration:

(i) Forward Integration:

$${}_{-\infty}W_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-u)^{\alpha-1} f(u)du$$

(ii) Backward Integration:

$${}_x W_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (u-x)^{\alpha-1} f(u)du$$

B. Fractional derivatives

(a)Riemann- Liouville left hand definition (RL LHD):
Let us select an integer m>α, α: fractional number such that

i) Integrate the function (m-α)- folds in the sense of forward RL
ii) Differentiate the above result by m

Then, RL LHD is defined as

$${}_0 D_t^{\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(u)}{(x-u)^{\alpha+1-m}}du\right]$$

(m-1≤α<m)

(b)Caputo right hand definition (Caputo RHD)
Let us select an integer m>α, α: fractional number such that

(i) Differentiate the function m times
(ii) Integrate the above result (m-α)- fold by RL LHD integration method.

Thus, Caputo's RHD is defined as

$${}_0^C D_t^{\alpha} = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\frac{d^m}{dt^m} f(u)}{(t-u)^{\alpha+1-m}}du \quad (m\text{-}1\le\alpha<m)$$

(c) Fractional derivatives other than Caputo and RL:

$${}_0^{\beta} D_t^{\alpha} = {}_0 I_t^{\beta(1-\alpha)} \frac{d}{dt}\left[{}_0 I_t^{(1-\beta)(1-\alpha)} f(t)\right]$$

For β=0, we have the α^{th} order derivative in the sense of RL LHD with m=1.
For β=1, we have the α^{th} order derivative in the sense of Caputo RHD with m=1.

(d) Fractional differintegral Grunwald- Letnikov (GL):

$${}_a D^{\alpha} f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{m=0}^{\left[\frac{x-a}{h}\right]} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x-mh),$$

$\left[\frac{x-a}{h}\right]$: floor function.

(e) Weyl's derivative:
For m-1≤α<m where m∈N,

$${}_x D_{\infty}^{\alpha} f(x) = (-1)^m \frac{d^m}{dx^m}\,{}_x W_{\infty}^{\alpha} f(x),$$

where ${}_x W_{\infty}^{\alpha} f(x)$: Weyl's fractional integration of order α.

It is to be noted that the RL LHD, Caputo RHD and GL are all equivalent.

III. GENERALIZED DYNAMIC SYSTEM AND ITS SOLUTION USING ADOMAIN DECOMPOSITION METHOD (ADM)

The general dynamic system may be described as

$$Fu = G \qquad (1)$$

where, F: general linear or non-linear differential equation which may be fractional in the sense of Riemann-Liouville. This operator can be decomposed as,

$$Fu = L_0 + Ru + NU = G \qquad (2)$$

L_0: highest order derivative (may be positive integer or fractional order), which is invertible i.e.

$L_0 u = \frac{d^m u(x)}{dx^m} = D_x^m u(x)$, which is linear operator.

R: linear differential (remainder) operator of order less than that of L_0; this can also be fractional linear operator i.e.

$$Ru \equiv a_1 D_x^{m-1} + a_2 D_x^{m-2} + a_3 D_x^{m-3} + ... + a_{m-1} D_x^1$$

, where $a_0, a_1, a_2, \ldots, a_{m-1}$ are constants.

N: non- linear part which will be decomposed into infinite sum of adomain polynomial (this term may be linear or constant) i.e.

$Nu=a_m u(x)+b_k[u(x)]^k+b_{k-1}[u(x)]^{k-1}+b_{k-2}[u(x)]^{k-2}+....+b_0 u(x)$

G(x): sum of all external stimulus source/ sink.

The R and N are internal stimuli and when it get balanced with the external stimulus G then the process of the parameter remains static without any growth or decay. Otherwise, the process parameter will have a solution as infinite (or finite) decomposed modes; generated by the system itself to oppose stimulus generated internally by previous mode.

The idea of solution to (1) is a new approach called Adomain Decomposition Method (ADM). ADM method is an analytical method and has a certain advantage over standard numerical techniques. The decomposed parts of ADM method are related to system reactions of various modes from zeroth mode to infinity mode. The sum of all these modes is the solution of (1).

In (2), if L_0 is the fractional differential operator in the sense of RL, then the initial conditions should be in fractional order say, $u^m(0)$, $u^{m-1}(0)$ for m to be fractional. These states are hard to visualize. But with this ADM, the RL formulation does not need these fractional initial states instead requires u(0), $u^{(1)}(0)$, $u^{(2)}(0)$ etc.; the integer order states gives the solution and thus physically understandable.
Equation (2) can be rewritten as

$L_0u=G-Ru-Nu$

Applying invertible operator both sides, we get

$$u = \Phi + L_0^{-1}G - L_0^{-1}[R(u)] - L_0^{-1}[N(u)] \quad (3)$$

Φ: solution of homogeneous equation $L_0u=0$, so that $L_0\Phi=0$ and this comes from initial and boundary condition.

Then with ADM, we have,
$u(\lambda)=u_0+\lambda u_1+ \lambda^2 u_2+ \lambda^3 u_3+…….$

This is a Maclaurin's series with respect to λ with coefficient u_n (n=1,2,…..) and $u_n=u^{(n)}(0)/n!$ around λ=0. Then, N(u) is a Maclaurin's series with respect to λ and obtain

$$N(u) = \sum_{n=0}^{\infty} \lambda^n A_n ,$$

where $A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]$.

For N(u) linear, $A_0=u_0$ and $A_n=u_n$.
For N(u) to be non- linear, $A_0=N(u_0)$,

$$A_1 = u_1 N'(u_0), A_2 = u_2 N'(u_0) + \frac{1}{2}u_1^2 N''(u_0)$$

$$, A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{u_1^3}{3!} N'''(u_0)$$

etc.
The parameter λ is just an identifier for collecting terms in suitable way, such that u_n depends on u_0, u_1, u_2,….,u_{n-1} and later on we will set λ=1. Parameterizing the equation (3), we get,

$$u = \Phi + L_0^{-1}G - \lambda L_0^{-1}[R(u)] - \lambda L_0^{-1}[N(u)] \quad (4)$$

Expanding the decomposition, (4) gives

$$u = \Phi + L_0^{-1}G - \lambda L_0^{-1}R\left(\sum_{n=0}^{\infty}\lambda^n u_n\right) - \lambda L_0^{-1}N\left(\sum_{n=0}^{\infty}\lambda^n A_n\right) \quad (5)$$

Comparing the like powers of λ in the expression for n=0, we have,
$u_0=\Phi+ L_0^{-1}G$. Similarly, for n=1,2,…, we have

$$u_n = -L_0^{-1}R(u_{n-1}) - L_0^{-1}(A_{n-1}) \text{ for } n\geq 1 \quad (6)$$

IV. EXAMPLE PROBLEM OF FRACTIONAL ORDER DIFFERENTIAL EQUATION (THIRD ORDER MASS SPRING SYSTEM WITH HALF ORDER ELEMENT)

Let us consider a fractional oscillator of third order system (mass spring system) with half order element described by

$$\frac{d^3x}{dt^3} + {}_0D_t^{1/2}x(t) + +x(t) = f(t) \quad (7)$$

with $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$ and f(t)=δ(t) (force term) and ${}_0D_t^{1/2}x(t)$ is the fractional order derivative in the sense of Riemann-Liouville (RL).

Solution of (7) using Adomain Decomposition Method (ADM):
Equation (7) can be rewritten as Fx= G,

Where $Fx=L_0x+Rx+Nx$ and G= δ(t) and $L_0= D_t^3$, R= $D_t^{1/2}$, N=x (linear).

As mentioned in section 3, the ADM method generates the mode as follows:
$x_0 = \Phi + L_0^{-1}G$, where Φ is the solution of homogeneous equation $L_0u=0$, so that $L_0\Phi=0$.

Thus, $\Phi = x(0) + t\dot{x}(0) + t^2\ddot{x}(0) = 0$.

So, $x_0 = L_0^{-1}G = D_t^{-3}\delta(t) = \frac{t^2}{2}$.

$A_0=x_0$ (as N is linear)

$$x_1 = -L_0^{-1}R(x_0) - L_0^{-1}A_0 = -(D_t^{-5/2} + D_t^{-3})x_0 = -\left(\frac{t^{9/2}}{\Gamma(11/2)} + \frac{t^5}{\Gamma(6)}\right)$$

TABLE I. THE MODAL FORCE AND DISPLACEMENT FOR THIRD ORDER SYSTEM WITH FRACTIONAL ORDER (HALF ORDER) DAMPING

Mode	Force	Displacement
0	δ(t)	$t^2/2$
1	$-t^2/2$	$-\left(\frac{t^{9/2}}{\Gamma(11/2)}+\frac{t^5}{\Gamma(6)}\right)$
2	$\left(\frac{t^{9/2}}{\Gamma(9/2)}+\frac{t^5}{\Gamma(6)}\right)$	$\left(\frac{t^7}{\Gamma(8)}+2\frac{t^{15/2}}{\Gamma(17/2)}+\frac{t^8}{\Gamma(9)}\right)$
3	$-\frac{t^7}{\Gamma(8)}-2\frac{t^{15/2}}{\Gamma(17/2)}-\frac{t^8}{\Gamma(9)}$	$-\left(\frac{t^{19/2}}{\Gamma(21/2)}+3\frac{t^{10}}{\Gamma(11)}+3\frac{t^{21/2}}{\Gamma(23/2)}+\frac{t^{11}}{\Gamma(12)}\right)$
……	……	……

using Riemann- Liouville (RL) integration $D^{-\alpha}f(t)=I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}f(u)du$

where, α is fractional.

$A_1= x_1$

$$x_2=-L_0^{-1}R(x_1)-L_0^{-1}A_1=-(D_t^{-5/2}+D_t^{-3})x_1=\left(\frac{t^7}{\Gamma(8)}+2\frac{t^{15/2}}{\Gamma(17/2)}+\frac{t^8}{\Gamma(9)}\right)$$

$A_2=x_2$

$$x_3=-L_0^{-1}R(x_2)-L_0^{-1}A_2=-(D_t^{-5/2}+D_t^{-3})x_2=-\left(\frac{t^{19/2}}{\Gamma(21/2)}+3\frac{t^{10}}{\Gamma(11)}+3\frac{t^{21/2}}{\Gamma(23/2)}+\frac{t^{11}}{\Gamma(12)}\right)\text{etc.}$$

Therefore, the solution of (7) is given by,

$$x(t)=\frac{t^2}{\Gamma(3)}-\frac{t^{4.5}}{\Gamma(5.5)}-\frac{t^5}{\Gamma(6)}+\frac{t^7}{\Gamma(8)}+2\frac{t^{7.5}}{\Gamma(8.5)}+\frac{t^8}{\Gamma(9)}-\frac{t^{9.5}}{\Gamma(10.5)}-3\frac{t^{10}}{\Gamma(11)}-3\frac{t^{10.5}}{\Gamma(11.5)}-\frac{t^{11}}{\Gamma(12)}.+\ldots$$

V. THIRD ORDER RC CIRCUIT WITH SEMI-INFINITE LOSSY CABLE ACTS AS HALF ORDERELEMENT CONNECTED TO A CRO PROBE

The physics of the above problem may be viewed as RC circuit reacting to an impulse current reaction in the presence of semi-infinite RC cable (CRO probe) connected to a shunt to measure the current. The semi-infinite lossy cable acts as half order element and third order circuit reaction is modified with the presence of half order element. In other words, the current into the line is equal to the half derivative of applied voltage. Practically, i(t) be instantaneous current of RC circuit connected to the battery by a switch. The consecutive equation for the circuit is described by

$$\frac{d^3}{dt^3}i(t)+\frac{1}{R}D_t^{1/2}i(t)+\frac{1}{RC}i(t)=\frac{V_{BB}}{R}\delta(t) \qquad (8)$$

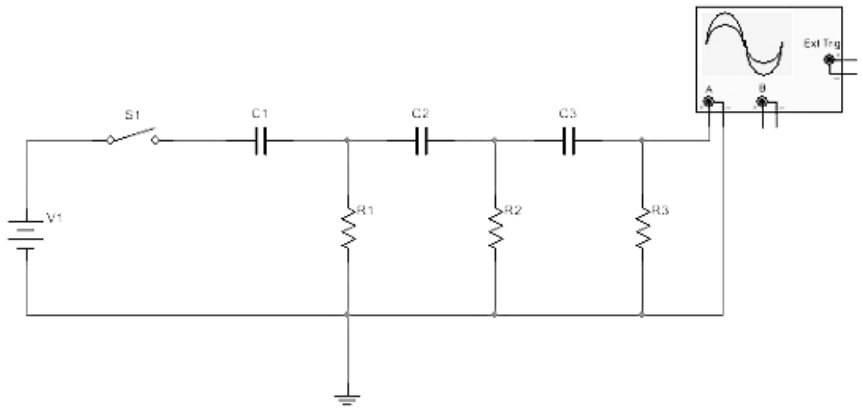


Figure 1. Third order RC circuit with semi- infinite cable as fractional half order element connected to a CRO probe

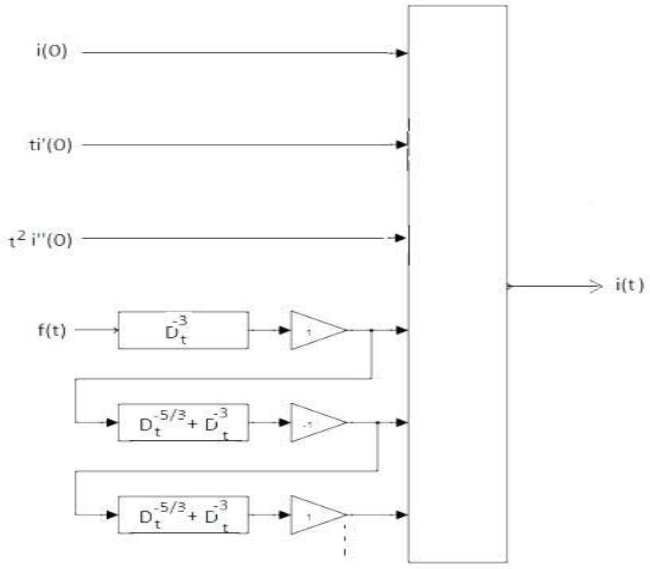


Figure 2. Block diagram showing solution of thirdorder differential equation by decomposition in presence of fractional order term

$i(0) = i'(0) = i''(0) = 0$. This is demonstration of oscillator (RC circuit) with fractional order element (half order) shown in Figure 1. Thus, the basic equation (8) gives the rate of change of current as related to the external stimulus. It is therefore demonstrated that ADM is actually translates the physics of the process where any change is opposed by the system itself. The system transfer function for analysis is a delta function (forcing function). However, the functions like Heaviside's step, Ramp or Sinusoidal function is obtained by convoluting this delta function response with other excitation function.

Using ADM, the solution of (8) with block diagram (see Figure 2) is given by $i(t) = i_0 + i_1 + i_2 + i_3 + ...$

$$= \frac{V_{BB}}{R}\left[\frac{t^2}{\Gamma(3)} - \frac{1}{R}\frac{t^{4.5}}{\Gamma(5.5)} - \frac{1}{RC}\frac{t^5}{\Gamma(6)} + \frac{1}{R^2}\frac{t^7}{\Gamma(8)} + \frac{2}{R^2C}\frac{t^{7.5}}{\Gamma(8.5)} + \frac{1}{R^2C^2}\frac{t^8}{\Gamma(9)} - \frac{1}{R^3}\frac{t^{8.5}}{\Gamma(9.5)} - \frac{3}{R^3C}\frac{t^{10}}{\Gamma(11)} - \frac{3}{R^3C^2}\frac{t^{10.5}}{\Gamma(11.5)} - \frac{1}{R^3C^3} + ...\right]$$

CONCLUSION

The main goal of this paper is to explore the fractional calculus especially fractional order differential equation in an engineering problems viz., a third orderRC circuit. It is generally represented by circuit equation of integer order. But in reality, the current into the line is equal to the half derivative of applied voltage. This leads to a fractional differential equation with integer order RC circuit with fractional order element. Now, in order to tackle the solution to the system of integer order, a series solution can be decomposed as Maclaurin's series at origin with all derivatives components. Similar procedure is applied for fractional order differential equation using ADM approach, which generates the infinite modes of reaction and therefore close to physical reality.

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