SOME FIXED POINT THEOREM IN RbMS FOR WELL KNOWN CONTRACTION

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Abstract— The aim of this paper is to obtain a fixed point theorem in the generalization of rectangular metric space and b-metric space. Our result extends some well-known fixed point theorem given by Reich and many others in the literature.

Keywords— Fixed Point, b-Metric space, Ractangular b-Metric space.

I. INTRODUCTION

Fixed point theory is one of the most dynamic research subject in nonlinear analysis. In 1922, Banach established a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by many authors for single-valued and multi-valued operators (see, e.g. [8], [9], and [12]). Bakhtin [1] and Czerwik [6] introduced the concept of b-metric space. In [6], Czerwik proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then many research papers have dealt with fixed point theory in such space (see, e.g. [3], [4], and [5]).Branciari [2] introduced the concept of rectangular metric space and proved an analogue of Banach contraction principle in such space. Recently, George et al. [7] introduced the concept of rectangular b-metric space as a generalization of metric space, rectangular metric space, and b-metric space and proved an analogue of Banach contraction principle and Kannan's fixed point theorem in such space.In this paper, we present a fixed point theorem which extend the theorem given by Reich [12] to rectangular b-metric space.

II. PRELIMINARIES

2.1. Definition.

Let X be a nonempty set and the mapping $d: X \times X \to [0,\infty)$for all $u, v, w \in X$, satisfies:
(b1) $d(u,v) = 0$ if and only if $u = v$;
(b2) $d(u,v) = d(v,u)$;
(b3)there exist areal number $s \geq 1$ such that $d(u,w) = s[d(u,v) + d(v,w)]$.
Then$d$is called a b-metric on $X$and (X,d) is called a b-metric space (in short bMS).(See in [10].)

From the above definition the class of b-metric spaces is larger than the class of (usual) metric spaces. For $s = 1$ it reduces into (usual) metric space. However, Czerwik [6] have shown that a b-metric on X need not to be a (usual) metric on X.

2.2. Definition.

Let X be a nonempty set and the mapping $d: X \times X \to [0,\infty)$satisfies:
(R1) $d(u,v) = 0$ if and only if $u = v$
for all $u, v \in X$;
(R2) $d(u,v) = d(v,u)$ for all $u, v \in X$;
(R3) $d(u,v) = d(u,x) + d(x,y) + d(y,v)$
for all $u, v \in X$ and all distinct points $x, y \in X\backslash\{u,v\}$.
Then d is called a rectangular metric on X and (X,d) is called a rectangular metric space (in short RMS). (See in [2].)

2.2. Definition.

Let X be a nonempty set and the mapping $d: X \times X \to [0,\infty)$satisfies:
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(Rb2) $d(u,v) = d(v,u)$ for all $u, v \in X$;
(Rb3) there exist areal number $s \geq 1$ such that $d(u,v) = s[d(u,x) + d(x,y) + d(y,v)]$
for all $u, v \in X$ and all distinct points $x, y \in X\backslash\{u,v\}$.
Then d is called a rectangular b-metric on X and (X,d) is called a rectangular b-metric space (in short RbMS). (See in [7].)

Note that, limit of a sequence in a RbMS is not necessarily unique and also every convergent sequence in a RbMS is not necessarily a Cauchy sequence.(See in [7].)

2.3. Example.

Let $X = \mathbb{N}$, define $d: X \times X \to X$ by

$$d(u,v) = \begin{cases} 0, & if\ u = v; \\ 4\alpha, & if\ u, v \in \{1,2\} and\ u \neq v; \\ \alpha, & if\ u\ or\ v \notin \{1,2\} and\ u \neq v, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but $(X,d)$is not a rectangular metric space, as $d(1,2) = 4\alpha > 3\alpha = d(1,3) + d(3,4) + d(4,2)$.(See in [7].)

2.4. Example.

Let $X = \mathbb{N}$, define $d: X \times X \to X$ by

$$d(x,y) = \begin{cases} 0, & if u = v; \\ 10\alpha, & if\ u = 1, v = 2; \\ \alpha, & if\ u \in \{1,2\}\ and\ v \in \{3\}; \\ 2\alpha, & if\ u \in \{1,2,3\} and\ v \in \{4\}; \\ 3\alpha, & if\ u\ or\ v \notin \{1,2,3,4\} and\ u \neq v, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = 2 > 1$, but (X,d) is not a rectangular metric space, as $d(1,2) = 10\alpha > 5\alpha = d(1,3) + d(3,4) + d(4,2)$.

Note that every b-metric space with coefficient s is a RbMS with coefficient s^2 but the converse is not necessarily true. (See in [7].)

2.5. Definition.

Let (Y,d) be a rectangular b-metric space. Then a sequence $\{y_n\}$in$Y$ is called convergent sequence if and only if there exists $y \in Y$, such that for every $\varepsilon > 0$, there exists $n(\varepsilon) \in N$, such that $d(y_n, y) < \varepsilon$ for all $n \geq n(\varepsilon)$. In this case we write $\lim_{n\to\infty} y_n = y$.(See in [7].)

2.6. Definition.

Let (Y,d) be a rectangular b-metric space. Then a sequence $\{y_n\}$ in Y is called Cauchy sequence if every $\varepsilon > 0$, there exists $n(\varepsilon) \in N$, such that $d(y_n, y_m) < \varepsilon$ for all $m, n \geq n(\varepsilon)$.(See in [7].)

2.7. Definition.

The rectangular b-metric space (Y,d) is complete if every Cauchy sequence in $Y$is convergent sequence in Y.(See in [7].)

2.8. Definition.

Let Xbe any set and $T: X \to X$ a selfmap. We say that $x \in X$is a fixed point of T if $T(x) = x$, for the convenience we write $Tx = x$. Recall $T^n x$ the nth iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, n = 1,2,3,\ldots \quad (2.1)$$

Is called the sequence of successive approximations with the initial value x_0.It is known as the Picard iteration starting atx_0.(See in [10].)

III. MAIN RESULT

3.1. Theorem.

Let (X,d) is a complete rectangular b-metric space with constants$s > 1$. Let $T: X \to X$satisfying:

$d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y)$(3.1)

for all $x, y \in X$, where $a, b, c$are non-negative real numbers and satisfy $a + s(b + c) < 1$. Then there exists a unique fixed point.

Proof. Let $x_0 \in X$and $\{x_n\}$ be a sequence in X define as (2.1). If $x_n = x_{n+1}$ then x_n is fixed of T. Suppose that $x_n \neq x_{n+1}$ for all$n \geq 1$.

Setting $\rho_n = d(x_n, x_{n+1})$ for all $n \geq 1$. It follows from the definition of RbMS, (2.1) and (3.1) that

$$\begin{aligned} &d(x_{n+1}, x_n) \\ &= d(Tx_n, Tx_{n-1}) \\ &\leq a\, d(x_n, Tx_n) + b\, d(x_{n-1}, Tx_{n-1}) + c\, d(x_n, x_{n-1}) \\ &= a\, d(x_n, x_{n+1}) + b\, d(x_{n-1}, x_n) + c\, d(x_n, x_{n-1}). \end{aligned}$$

So, we have

$d(x_{n+1}, x_n) \leq \left(\frac{b+c}{1-a}\right) d(x_n, x_{n-1})$. Let $\mu = \left(\frac{b+c}{1-a}\right)$,we can rewrite that $d(x_{n+1}, x_n) \leq \mu d(x_n, x_{n-1})$.It follows that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \mu d(x_n, x_{n-1}) \leq \mu^2 d(x_{n-1}, x_{n-2}) \leq \\ &\leq \mu^n d(x_1, x_0). \end{aligned}$$

Continuing this way, we obtain that$d(x_{n+1}, x_n) \leq \mu^n d(x_1, x_0) = \mu^n \rho_0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$. If m is even, we say that $m = 2q$ for integer $q \geq 1$. Consider,

$$\begin{aligned} &d(x_n, x_m) \\ &= d(x_n, x_{n+2q}) \\ &\leq s[(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + cd(x_n, x_{n+2q})] \\ &\leq s(\rho_n + \rho_{n+1}) + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + cd(x_{n+4}, x_{n+2q})] \\ &\leq s(\rho_n + \rho_{n+1}) + s^2(\rho_{n+2} + \rho_{n+3}) \\ &\quad + s^3(\rho_{n+4} + \rho_{n+5}) + \cdots \\ &\quad + s^{q-1}(\rho_{2q-4} + \rho_{2q-3}) \\ &\quad + s^{q-1}(\rho_{n+2q-2} + \rho_{n+2q}) \\ &\leq s(\mu^n \rho_0 + \mu^{n+1} \rho_0) + s^2(\mu^{n+2} \rho_0 + \mu^{n+3} \rho_0) \\ &\quad + s^3(\mu^{n+4} \rho_0 + \mu^{n+5} \rho_0) + \cdots \\ &\quad + s^{q-1}(\mu^{2q-4} \rho_0 + \mu^{2q-3} \rho_0) + s^{q-1}(\mu^{n+2q-2} \rho_0^*) \\ &\leq s\mu^n(1 + s\mu^2 + s^2\mu^4 + \cdots)\rho_0 \\ &\quad + s\mu^{n+1}(1 + s\mu^2 + s^2\mu^4 + \cdots)\rho_0 \\ &\quad + s^{q-1}\mu^{n+2q-2}\rho_0^* + s^{q-1}\mu^{n+2q-2}\rho_0^* \\ &= s(\mu^n \rho_0 + \mu^{n+1} \rho_0) + s^2(\mu^{n+2} \rho_0 + \mu^{n+3} \rho_0) \\ &\quad + s^3(\mu^{n+4} \rho_0 + \mu^{n+5} \rho_0) + \cdots \\ &\quad + s^{q-1}(\mu^{2q-4} \rho_0 + \mu^{2q-3} \rho_0) + s^{q-1}(\mu^{n+2q-2} \rho_0^*). \end{aligned}$$

It follows that

$$\begin{aligned} &d(x_n, x_m) = d(x_n, x_{n+2q}) \\ &\leq \frac{1+\mu}{1-s\mu^2}(s\mu^n)\rho_0 + s^{q-1}\mu^{n+2q-2}\rho_0^* \\ &\leq \frac{1+\mu}{1-s\mu^2}(s\mu^n)\rho_0 + (s\mu)^{2q}\mu^{n-2}\rho_0^* \\ &\leq \frac{1+\mu}{1-s\mu^2}(s\mu^n)\rho_0\mu^{n-2}\rho_0^*. \end{aligned}$$

Hence, If m is even we have

$$\lim_{n\to\infty} d(x_n, x_m) = 0. \quad (3.2)$$

Again, If m is odd, we say $m = 2q + 1$ for integer $q \geq 1$. Consider,

$$\begin{aligned} &d(x_n, x_m) = d(x_n, x_{n+2q+1}) \\ &\leq s[(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + d(x_{n+2}, x_{n+2q+1})] \\ &\leq s(\rho_n + \rho_{n+1}) + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2q+1})] \\ &\leq s(\rho_n + \rho_{n+1}) + s^2(\rho_{n+2} + \rho_{n+3}) \\ &\quad + s^3(\rho_{n+4} + \rho_{n+5}) + \cdots + s^q \rho_{n+2q} \\ &\leq s(\mu^n \rho_0 + \mu^{n+1} \rho_0) + s^2(\mu^{n+2} \rho_0 + \mu^{n+3} \rho_0) \end{aligned}$$

$$+s^3(\mu^{n+4}\rho_0 + \mu^{n+5}\rho_0) + \cdots + s^q\mu^{n+2q}\rho_0$$
$$\leq s\mu^n(1 + s\mu^2 + s^2\mu^4 + \cdots)\rho_0$$
$$+s\mu^{n+1}(1 + s\mu^2 + s^2\mu^4 + \cdots)\rho_0$$
$$= \frac{1+\mu}{1-s\mu^2}(s\mu^n)\rho_0 \text{ (where } s\mu^2 < 1).$$

It follows that

$$d(x_n, x_m) = d(x_n, x_{n+2q+1}) \leq \frac{1+\mu}{1-s\mu^2}(s\mu^n)\rho_0.$$

Hence, If m is odd we have

$$\lim_{n\to\infty} d(x_n, x_m) = 0. \qquad (3.2)$$

We have from (3.2) and (3.3) that for all $m, n > 0$ with $m > n$,

$$\lim_{n\to\infty} d(x_n, x_m) = 0. \qquad (3.3)$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Now, we show that z is a fixed point of T. For any $n \in \mathbb{N}$ we have

$$d(z, Tz) \leq s[d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, T_n z)]$$
$$\leq s[d(z, x_n) + d_n + d(Tx_n, Tz)]$$
$$\leq s[d(z, x_n) + d_n + \mu d(x_n, z)].$$

Taking limit as $n \to \infty$and using (3.3), we obtain $d(z, Tz) = 0$, i.e., $Tz = z$. Hence, z is a fixed point of T. Now we show that u is the unique fixed point of T. Assume that w is an other fixed point of T. Then we have $Tw = w$. Then it follows from (3.1) that

$$d(z, w) = d(Tz, Tw)$$
$$\leq ad(z, Tz) + bd(w, Tw) + cd(z, w)$$
$$\leq cd(z, w),$$

which is a contradiction. So, we have $z = w$. Therefore, T has a unique fixed point. It completes the proof.

If we reduce our theorem to bMS, we have the followings theorems, Reich's type fixed point theorem which given by Pankaj et al. [11] and Kannan's type fixed point theorem which given by Mehmet and Kiziltunc [10], respectively.

3.2. Theorem.

Let (X, d) is a complete b-metric space with constants $s > 1$. Let $T: X \to X$ satisfying:

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y)$$

for all $x, y \in X$, where a, b, c are non-negative real numbers and satisfy $a + s(b + c) < 1$. Then there exists a unique fixed point.

3.3. Theorem.

Let (X, d) is a complete b-metric space with constants $s > 1$. Let $T: X \to Xbe$ a mapping for which there exists $\mu \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then there exists a unique fixed point.

Moreover, our result can be reduced to the following theorem, Kannan's type fixed point theorem in RbMSwhich given by [7]and reference therein.

3.4. Theorem.

Let (X, d) be a complete rectangular b-metric space with coefficients $s > 1$. Let $T: X \to X$ satisfying:

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\mu \in [0, 1/s + 1)$. Then T has a unique fixed point.

CONCLUSIONS

Our main result show that there exists a unique fixed point in RbMS which is the generalization of metric space, rectangular metric space, and b-metric space for some well-known contraction as Reich'stype and Kannan's type. It also extend theorem 3.2about Reich's type fixed point in bMS ofPankajet al. in [11] and theorem 2 about Kannan type fixed point in bMS of Mehmet in [10].

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