SOME FIXED POINT FOR TWO PAIRSCOMPATIBLE MAPS

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Abstract— In this paper, our purpose is to show that two pairs of compatible mappings have unique common fixed point in b-metric space which generalizes and extends the similar results in the literature.

Keywords— Fixed Point, Metric space, b-Metric space, Compatible Mapping.

I. INTRODUCTION

Fixed point theory has many applications in various branches of Mathematics and branch of science. One of the most important results of the mathematical analysis is the well-known Banach fixed point theorem proved by Banach in 1922. Since then, Manyresearch papers have dealt with fixed point theory of single-value and multivalued operators (see, e.g. ref.2, 7, 8, 9, 10, and 14).The concept of a b-metric space was introduced by Czerwik in (ref.3) and Bakhtin in (ref.1), which a generalized of (usual) metric space.Since the publication of these works, several papers have appeared which were concerned to these spaces. Pankaj et al. (ref.13)gave some results related fixed point theorem in b-metric spaces. They have shown the extension theorem given by Reich (ref.15) and Hardy and Rogers (ref.6) to the b-metric spaces. Otherwise, Mehmet et al. (ref.12)gave some results for contraction in b-metric space. Recently, Dubey, Shukla, and Dubey in (ref.5), Malhotra and Bansal (ref.11) study the extension of fixed point theorems in b-metric space. In this paper, we present a common fixed point theorem for two pairs of mappings of compatible type in b-metric spaces which generalize and improve similar results of fixed point in our main result.

II. PRELIMINARIES

2.1. Definition.

Let X be a nonempty set and $s \geq 1$ a given real number. A function $d: X \times X \to R^{+}$(nonnegative real numbers) is called a b-metric provided that for all $x, y, z \in X$, the following conditions are satisfied

1) $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z) = s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space. We know from definition that a metric space is evidently a b- metric space for $s = 1$. (See ref. 12)

The following show that (X, d) is a b-metric space.

2.2. Example.

Let $X = \{0,1,2\}$ and $d(2,0) = d(0,2) = m \geq 2, d(0,1) = d(1,2) = d(2,1) = 1$ and $d(0,0) = d(1,1) = d(2,2) = 0$. Then $d(x, y) \leq \frac{m}{2}[d(x, 2) + d(2, y)]$ for all $x, y, z \in X$. If $m > 2$, the ordinary triangle inequality does not hold.(See ref. 11)

2.2. Example.

The space $L_p(0 < p < 1)$ of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p < \infty$, is b-metric space if we take$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}$,for each $x, y \in L_p$.(See ref. 13)

2.3. Definition.

Let(X, d) is a b-metric space. Then a sequence $\{x_n\}$ in X is called Cauchy sequence if and only if for every $\varepsilon > 0$, there exists $n(\varepsilon) \in N$, such that for all $m, n \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.(See ref. 12)

2.4. Definition.

Let (X, d) is a b-metric space. Then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$, such that for every $\varepsilon > 0$, there exists $n(\varepsilon) \in N$, such that $d(x_n, x) < \varepsilon$ for all $n \geq n(\varepsilon)$. In this case we write $\lim_{n\to\infty} x_n = x$.(See ref. 13)

2.5. Definition.

The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.(See ref. 12)

2.5. Definition.

Two self-maps S and T of a nonempty set X are called to be commuting if $STx = TSx$ for all $x \in X$. (See ref. 11)

2.6. Definition.

Two self-maps S and T of a nonempty set $X$are called compatible if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$

III. MAIN RESULTS AND DISCUSSION

3.1. Theorem.

Let (X, d) is a b-metric space with constant $s \geq 1$. Let $G, K, S, T: X \to X$ be mappings such that $T(X) \subset G(X), S(X) \subset K(X)$ and two

pairs$(T,K),(S,G)$ are compatible and commuting pair of maps. If $d(Tx,Sy) \le \alpha d(Kx,Gy) + \beta[d(Gy,Sy) + d(Kx,Tx)]$ holds for each $x,y \in X$ with $\alpha + 2\beta < 1$, then G,K,S, and T have a unique common fixed point.

Proof. Let $x_0 \in X$. As $T(X) \subset G(X)$ there exist $x_1 \in X$ such that $Tx_0 = Gx_1$. Since $S(X) \subset K(X)$and $Sx_1 \in K(X)$, we can choose $x_2 \in X$ such that $Sx_1 = Kx_2$. In general, we define a sequence $\{z_n\} \in X$such that $z_{2n} = Tx_{2n} = Gx_{2n+1}$ and $z_{2n+1} = Sx_{2n+1} = Kx_{2n+2}$for all $n \ge 0$. Let us consider

$$\begin{aligned} d(z_{2n},z_{2n+1}) &= d(Tx_{2n},Sx_{2n+1}) \\ &\le \alpha d(Kx_{2n},Gx_{2n+1}) \\ &\quad + \beta d(Gx_{2n+1},Sx_{2n+1}) \\ &\quad + \beta d(Kx_{2n},Tx_{2n}) \\ &= \alpha d(z_{2n-1},z_{2n}) \\ &\quad + \beta d(z_{2n},z_{2n+1}) \\ &\quad + \beta d(z_{2n-1},z_{2n}). \end{aligned}$$

It follows that $d(z_{2n},z_{2n+1}) \le qd(z_{2n-1},z_{2n})$ for all $n \ge 0$, where $0 < q = \frac{\alpha+\beta}{1-\beta} < 1$. Continuing this way we obtain $d(z_{2n},z_{2n+1}) \le qd(z_{2n-1},z_{2n}) \le \cdots \le q^n d(z_0,z_1)$. So that $d(z_n,z_{2n+1}) \le q^n d(z_0,z_1)$for all $n \ge 1$, where $0 < q < 1$. Now we show that $\{x_n\}$ is a Cauchy sequence in X. For every $m,n \in N$ such that $m > n$, we have

$$\begin{aligned} d(z_n,z_m) &\le sd(z_n,z_{n+1}) + s^2 d(z_{n+1},z_{n+2}) + \cdots \\ &\quad + s^{m-1} d(z_{m-1},z_m) \\ &\le [sq^n + s^2 q^{n+1} + \cdots \\ &\quad + s^{m-n} q^{m-1}] d(z_1,z_0) \\ &< [sq^n + s^2 q^{n+1} + \cdots] d(z_1,z_0) \\ &\le \frac{sq^n}{1-sq}\, d(z_1,z_0). \end{aligned}$$

Taking as$n,m \to \infty$, we have $\{z_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, there exists a point $z \in X$ such that $\lim_{n\to\infty} z_n = z$.Consequently subsequences, $\lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} Kx_{2n} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n} = z$. Next, we show that z is a common fixed point of G,K,S, and T. Since (T,K) is compatible and commuting, we have

$$\lim_{n\to\infty} d(KTx_{2n},TTx_{2n}) = \lim_{n\to\infty} d(TKx_{2n},TTx_{2n}) = 0.$$

Now, we show that z is the fixed point of T. Consider,

$$\begin{aligned} d(TTx_{2n},Sx_{2n+1}) &\le \alpha d(KTx_{2n},Gx_{2n+1}) \\ &\quad + \beta d(Gx_{2n+1},Sx_{2n+1}) \\ &\quad + \beta d(KTx_{2n},TTx_{2n}). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$d(Tz,z) \le \alpha d(Tz,z) + \beta d(z,z) + \beta(0).$$

Hence, we obtain$d(Tz,z) = 0$.We have$z$ is the fixed point of T. Next, we show that z is the fixed point of K.Again, we have

$$\begin{aligned} d(KTx_{2n},Sx_{2n+1}) &\le \alpha d(KTx_{2n},Gx_{2n+1}) \\ &\quad + \beta d(Gx_{2n+1},Sx_{2n+1}) \\ &\quad + d(KTx_{2n},TTx_{2n}). \end{aligned}$$

Taking limit as$n \to \infty$, we have

$d(Kz,z) \le \alpha d(Kz,z) + \beta d(z,z) + \beta(0)$,which is a contradiction. Hence, we have$d(Kz,z) = 0$.Thus, we have z is the fixed point of K.Similarly, since the pair (S,G) is compatible and commuting, we obtain

$\lim_{n\to\infty} d(GSx_{2n+1},SSx_{2n+1})$
$= \lim_{n\to\infty} d(SGx_{2n+1},GGx_{2n+1}) = 0$. Let us consider,

$$\begin{aligned} d(SSx_{2n+1},Tx_{2n}) &= d(Tx_{2n},SSx_{2n+1}) \\ &\le \alpha d(Kx_{2n},GSx_{2n+1}) \\ &\quad + \beta d(GSx_{2n+1},SSx_{2n+1}) \\ &\quad + \beta d(Kx_{2n},Tx_{2n}) \\ &\le \alpha d(Kx_{2n},SGx_{2n+1}) \\ &\quad + \beta d(GSx_{2n+1},SSx_{2n+1}) \\ &\quad + \beta d(Kx_{2n},Tx_{2n}). \end{aligned}$$

Taking limit as $n \to \infty$, we have $d(Sz,z) \le \alpha d(z,Sz) + \beta(0) + \beta d(z,z)$. We obtain $d(Sz,z) = 0$. Thus, we have z is the fixed point of S.Using the same result, we have z is the fixed point of G.Therefore, we have that z is a common fixed point of G,K,S, and T. Uniqueness. Let w is other common fixed point of G,K,S, and T. Then, we have

$$\begin{aligned} d(z,w) &= d(Tz,Sw) \\ &\le \alpha d(Kz,Gw) + \beta d(Gw,Sw) \\ &\quad + \beta d(Kz,Tz) \\ &\le \alpha d(z,w) + \beta d(w,w) \\ &\quad + \beta d(z,z). \end{aligned}$$

Which is a contraction. Thus G,K,S, and T have a common fixed point in X.This completes the proof.

3.2. Corollary.

Let (X,d) is a b-metric space with constant $s \ge 1$. Let $G,S,T: X \to X$ be mappings such that $T(X) \subset G(X), S(X) \subset G(X)$ and two pairs $(T,G),(S,G)$ are compatible and commuting pair of maps. If $d(Tx,Sy) \le \alpha d(Gx,Gy) + \beta[d(Gy,Sy) + d(Gx,Tx)]$ holds for each $x,y \in X$ with $\alpha + 2\beta < 1$, then G,S, and T have a unique common fixed point.

Proof. We put $K = G$ in the above theorem.We have thedesire result.

3.3. Corollary.

Let (X,d) is a b-metric space with constant $s \ge 1$. Let $G,S: X \to X$ be mappings such that$S(X) \subset G(X)$.The pair(S,G) are compatibleand commuting. If$d(Sx,Sy) \le \alpha d(Gx,Gy) + \beta[d(Gy,Sy) + d(Gx,Sx)]$holds for each $x,y \in X$ with $\alpha + 2\beta < 1$, then G and S have a unique common fixed point.

Proof. We put $K = G$ in the above theorem.We have thedesire result.

3.4. Corollary.

Let (X,d) is a b-metric space with constant $s \ge 1$. Let $I,S: X \to X$ be mappings such that $S(X) \subset X$and I is the identity map. The pair (S,I) compatibleand commuting. If $d(Sx,Sy) \le \alpha d(x,y) + \beta[d(y,Sy) + d(x,Sx)]$holds for each $x,y \in X$ with $\alpha + 2\beta < 1$, then$S$ has a unique common fixed point.

Proof. We put $K = G = I$ and $S = T$in the above theorem.We have the desire result.

CONCLUSIONS

In many papers have used the b-metric spaces in which b-metric function is continuous, but our theorem can be employed in the setup of discontinuous b-metric spaces. In 3.4 Corollary, we show that it is reduced from ourtheorem which is the result in theorem 3.1 of Pankaj et al.(ref. 13) and theorem 2 of Dubey et al.(ref. 5). On the other hand, our result can be reduced to the result in Theorem 2 of Mehmet et al. (ref. 12) and reference therein.

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