PORTFOLIO ALLOCATION WITH TAILDEPENDENCE CONSTRAINTS

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Abstract - This paper introduces tail dependence coefficient, which measures the comovement of a pair of random variables. In this case, random variables imply market and portfolio return. And the comovement of this pair measure probability that a portfolio return will be negative when the market. Moreover, this paper purposes specific model of market and portfolio return and model of dependency between portfolio and market to allocate a portfolio with tail dependence as a constraint. In addition, this paper would compare first the proportion of allocation between mean-variance and portfolio with tail dependence is? And second realized return of portfolio performance to show the best strategy.

Keywords - Correlation, Portfolio allocation, Student's t copula,Tail dependence

I. INTRODUCTION

Inrecent day, the concept of tail dependence has been studied widespreadin various finance fields. For instance, there are market index coupling, stock and market coupling, stock selection from tail dependence as a criterion, etc. This paper focuses on portfolio allocation using tail dependence. Before allocating portfolio, there are questions to answer in the following paragraphs. First, could it possible to adapt tail dependence coefficient to portfolio allocation,what assumption of stocks and market model that makes tail dependence concept available in the allocation. In the previous studies, the allocation has several models. For example, mean-variance portfolio, equal weight, market portfolio, etc. However, there is no model for a portfolio with tail dependence that is concerned. Thus, this paper proposes a new model of stocks and market for portfolio allocation under the tail dependence concept. After the model is defined, this paper would also investigate which variable effects portfolio tail dependence. The next research question is, the portfolio allocation which has tail dependence coefficient as a constraint, has a closed-form for calculation or not. If it has what is the closed-form. If it is not what method will be used to solve portfolio optimization with taildependenceconstraint. Then, answer the following question, how many kinds of solution possible, what is the message behind each solution and how it is different from the mean variance method. After that, to compare realized return between mean-variance and portfolio with tail dependence constraint. In addition, the portfolio with tail dependence constraint have volatility constraint as same as the mean-variance portfolio, then only the performance criterion is realized return.

In conclusion, the objective of this paper are answers of these following questions

1.What is the model of stocks and market return model and what is the copula model to study the portfolio tail dependence. Which variable is effective to portfolio tail dependence.

2.What is the closed-form of tail dependence coefficient as a constraint in portfolio optimization if the closed-form exist. If it is not which method is applied tail dependence to portfolio optimization as a constraint.

3.What is the difference between mean-variance and portfolio with tail dependence. How it is different in allocation? What is the implication of each strategy?

Before answeringthe first question, this paper looking forward to which type of copulas could find tail dependence coefficient by closed-form solution. Luckily, one of these types of copula is Student's t copula. Then the couple model of marginal stocks and the market could be Student's t. In addition, the marginal model of stocks and the market would be Student's t because, if an individual stock has approximately student's t distribution, it has a closure property which is the sum of random variables of the same distribution which is similar to marginal distribution. To answer the first question, each stock and market are coupled asStudent's t.Then, the weighting stocks and market are coupled as tcopula as well. After the marginal model and coupling are set, this study would investigate which variables affected portfolio tail dependence. In summary of the variables, the correlation between each stock with the market has linearly effects whichif the correlation of each stock is higher tail dependence of portfolio tail dependence is higher and vice versa. And for the portfolio volatility effect, if the portfolio volatility lowthen the tail dependence is high. Because the diversification is higher or similar to market.In the case of the market is crashing, this portfolio is likely to have the same effects. To reduce the crashing of portfolio when the market is crash, the tail dependence should be lower but the diversification benefit is reduced too. After the first question is answered, the second question is a consequence of the first. Since, the model of the return is student's t

distributed and the coupling is student's t which has closed-form of tail dependence coefficient. Then the closed-form of tail dependence constraint as a constraint in portfolio optimization is possible. Briefly, tail dependence as a constraint could have closed-formed in terms of weighting of the stock return model.

To answer the third question, the difference in the optimization to find optimal weight is a mean-variance portfolio with volatility constraint, but a portfolio with tail dependence have both volatility and tail dependence constraint. The solution of the mean-variance portfolio is maximizing return with diversification as much as possible or diversify to have less volatility as possible. In contrast, the solution of portfolio with tail dependence is diversify same as mean-variance portfolio. If the result of diversifying of mean-variance portfolio is not exceeded tail dependence constraint, the result of portfolio with tail dependence is also same as the mean variance solution. If the diversify is exceeded, portfolio with tail is trend to invest in one stock more than others or less diversify than mean variance portfolio. Moreover, the difference in realizing returns in the situation of the market return is unsatisfactory. A criterion of performance will be realized return when investment in the situation.

Finally, the contributions of this paper are the following reasons. First, concept of tail dependence constraint should have applied to the portfolio because if the market return is bad, portfolio could have not crashed in this situation. Second, tail dependence coefficient could have applied to portfolio optimization if the dependency of market and portfolio is Student's t copula. In addition, there is closed-form solution of the coefficient which could add in portfolio optimization in terms of weight of the stocks as changing variables.

Lastly, in the previous researches point that the mean-variance portfolio has a diversification benefit, but this paper shows that diversification benefit is not alwaysa good property. If the market has dissatisfied return of the portfolio with tail could avoid a bad return or have better returns than the market.

II. VARIABLES EFFECTS TO PORTFOLIO TAIL DEPENDENCE

Mathematical model of stocks and market returns.$X_i = \frac{Z_i\sigma_i}{\sqrt{\frac{Z_0^2}{n}}} + \mu_i$.

Where the index $i \in \{0,1,2,3\}$ if $i = 0$stands for market return and $i = 1,2,3$ stand for stock return which stock 1 is the highest correlation with the market and stock, number 2 is the medium correlation with -the market and stock 3 is the lowest correlation with the market respectively. And σ_i is volatility of each return. Next, the expected return of market and stock is μ_i. All of returns has a common factor which defined as square root of chi-square degree v divided by v. Lastly, the variables that we are going to study is hidden in the variable Z_i. This is standard normal random variable and each Z_i are dependent by correlation. Where ρ_{ij} is correlation between return of i^{th} and j^{th}if $i = j$ the correlation is 1.

After we define model of market and individual stock return, we are going to define portfolio return model. Let $w_1, w_2$and w_3 be weight on stock 1,2 and 3 respectively. And the portfolio return model is

$X_p = w_1X_1 + w_2X_2 + w_3X_3$.

Next, tail dependence coefficient in assumption of Student's has closed-form below,

$$\lambda = 2 - 2t_{V+1}\left(\sqrt{1+v}\sqrt{\frac{1-\rho}{1+\rho}}\right). \quad (1)$$

As you can see that the variable effects tail dependence is only correlation between 2 random variables. So, the variable that we have to study is correlation between market and portfolio. Since, we don't have closed-form of correlation between market and port. Therefore, we have to derive for correlation's closed-form. To begin the proof with, the definition of correlation between portfolio and market

$$\rho_p = \frac{\mathrm{cov}(X_p, X_0)}{\sqrt{\mathrm{Var}(X_p)\mathrm{Var}(X_0)}}.$$

We could get all elements for correlation above using a definition of covariance of 2 variables. Then, we will get closed-form of portfolio correlation as follows

$$\rho_p = \frac{w^T a}{\sqrt{w^T \Sigma w}}. \quad (2)$$

Let a is the vector in the form similar to the equation below.

$a = [\rho_{01}\sigma_1\ \rho_{02}\sigma_2\rho_{03}\sigma_3]^T$.

To answer the first research questionof what variables,affect portfolio tail dependence. Begin with that the equation (1), which is tail dependence coefficient closed-form under thet-copula assumption, depends on the correlation between a variable with another variable. In this case, it is a correlation coefficient between the portfolio return and the market return. And this correlation is the equation (2).

Since, the equation contains these kinds of variables,there are correlations between each stock with market, portfolio volatility and weights of each stock. The effect of correlation between a stock and the market is direct variation to the correlation between portfolio with the market. If the correlation of stock with the market is high, the correlation between portfolio with the market trending to be high.

Next, the effect on portfolio volatility is inverse to portfolio correlation with the market. If the volatility is low, the correlation is high. If the volatility is high, the correlation is low.

Lastly, the effects of weights of stock have various results. The result depends on the states of weight, which is short or long. The portfolio correlation is the highest if a portfolio invests mostly in a stock, which has the highest correlation with the market with no short in the other stocks. If short position is shown, the portfolio correlation is low with dramatic rate of decrease. The portfolio correlation is lower if a portfolio invests mostly in a stock, which has the lowest correlation with the market, with short in the other stocks.

In conclusion, portfolio correlation varies on the correlation between stocks and market, portfolio volatility and weights. These variable effects lead to portfolio correlation. And portfolio tail dependence is the outcome of portfolio correlation. If portfolio correlation is high, portfolio tail dependence would high. And if portfolio correlation is low, portfolio tail dependence would lower.

III. PORTFOLIO OPTIMIZATION

After that, to find optimal weight of portfolio with tail dependence would be different form the weight of classical mean-variance portfolio. The optimization equation will be like the form below.

Maximizing $W^T E$ (3)

Subject to $\sqrt{W^T \Sigma W} \leq \sigma_p$ (4)

$f(w, \rho_p) \leq \lambda_p$. (5)

$W^T 1 = 1$. (6)

The objective function is maximizing return with a portfolio risk constraint that not exceed σ_p and tail dependence constraint that assumed tail dependence function assumed by student's t joint distribution as $f(w.\rho_p)$ and the threshold is λ_p. The rest of constraint or the equality constraint is summation of weight is equals to 1.

The left-hand side of equation (5) is the same as (1). It can re-write as

$$2 - 2t_{v+1}\left(\frac{\sqrt{1+v}\sqrt{1-\rho_p}}{\sqrt{1+\rho_p}}\right) \leq \lambda_p.$$

$$\frac{1}{\sqrt{1+v}} t_{v+1}^{-1}\left(1 - \frac{\lambda_p}{2}\right) \leq \sqrt{\frac{1-\rho_p}{1+\rho_p}}$$

For simplicity, let $\frac{1}{\sqrt{1+v}} t_{v+1}^{-1}\left(1 - \frac{\lambda_p}{2}\right)$ is x.

$$\rho_p \leq \frac{1-x^2}{1+x^2}.$$

For simplicity again by let y equals to $\frac{1-x^2}{1+x^2}$.And refers to equation (2) then the complete form of equation (5) is as follows

$W^T a - y\sqrt{W^T \Sigma W} \leq 0$ (5)

Therefore, the answer of the second research question is resolved. And the solution satisfied equations (3),(4),(5), and (6), it will be the optimal solution.

To answer the third research question, this section will show the result of optimization. This study would show the result of difference of portfolio allocation between mean-variance portfolio and portfolio with tail dependence. First, data of this research are the daily return of the Stock Exchange of Thailand or SET and the return of single stock that contains in SET100 or stock top 100 in SET. This daily return is covered from January, 2007 until December, 2008. The reason for the timing of the data is these years because in the year 2008 the market had bad return. If our assumption were right, a portfolio with tail dependence constraint should perform better than other portfolio that no tail dependence constraint.

Portfolios in this studyconsists of 3 stocks categorized by the level of correlation between each stock with the market. Stock selection would have these steps. First, find a correlation between each stock with a market, then ordering stocks by correlation from the highest to the lowest. Next, grouping stocks into 3 groups which are high, medium and low stock correlation. Finally, pick the median of each group. As a result of the selection, the representative for portfolio allocation are BANPU, EGCO and DELTA.

At that time, the SET index crashed. If an investor bought SET index in the early of July, 2008 until September, 2008, his portfolio would loss about 20%. And mean-variance portfolio from the representative have weighting in BANPU, EGCO and DELTA are 32.8%, 30.2% and 37%. Because, expected return of stocks are approximately equal. The allocation would be partially equal to get diversification benefit. In addition, portfolio with tail dependence have weighting in each stock are -20.8%, 59.6% and 61.2%. Because, invest in stock with less correlation with the market more than stock with positively strong correlation with market to get a portfolio with lower tail dependence than mean variance portfolio. The realized return of mean-variance portfolio and portfolio with tail dependence are -26% and -17% respectively.

In the same year, if that investor bought the index in October and sold in the late December, this investor would suffer his portfolio again with loss -29%. Alternatively, if the investment strategy was mean variance portfolio with the same representative,this portfolio would be weighting in BANPU, EGCO and DELTA 26.3%, 8% and 65.7% respectively. The reason were BANPU and DELTA have better expected return than EGCO. However, this ratio lead

portfolio tail dependence up to 33% which was considered too high. Therefore, portfolio with tail dependence would invest mostly in stock with weak correlation with marketin ratio of 89% in DELTA. And invest in EGCO 11%. As a result of this composition, tail dependence of portfolio with tail dependence is 20%. Finally, realized return of portfolios are -25% and -21%. If the investment portfolio is constrained by tail dependence, it could reduce the risk of tail dependence which is outperform mean variance portfolio when the market is crashing.

CONCLUSION

In this paper, the concept of tail dependence applies to portfolio allocation. In the previous study, a portfolio invested in each stock that have low tail dependence to the market. The older paper leads to a question that is it possible to allocate a portfolio with tail dependence constraint. Unfortunately, there is no model or constraint of tail dependence for portfolio allocation. Then this study has to purpose new model, stock and market return model, which is an approximately Student's t distribution, and dependency of returns is t copula. Because, Student's t copula has closed-form solution of tail dependence coefficient, the benefit of closed-form that available for tail dependence could analyze the variables that effect portfolio tail dependence.

Moreover, equation (10) shows that portfolio tail dependence is varies on the correlation between individual stock with the market, portfolio volatility and weight. The effects of correlation of stock with the market are linearly related. If the correlation is high, tail dependence of portfolio would be high directly the same and vice versa. Next, the impact of portfolio volatility is inverse. If the volatility is high, which is less diversification benefit, portfolio tail dependence is low, which is the portfolio is less likely replication to market. Lastly, the weighting of stock, if the weight of high correlation to market stock invests more, the portfolio tail dependence is higher. If there is a short sell happening, tail dependence is decreasing quickly. In addition, the equation would also be a constraint in portfolio optimization of portfolio with tail dependence. This paper would have allocated into 2 portfolios, which are classical mean-variance portfolio and portfolio with tail dependence. The difference of allocations is the mean-variance would have invested more in the stock with higher expected return and more diversify.

In contrast, portfolio with tail dependence is likely to invest with a high proportion in stock, which is less correlation to the market. As the result, if an investor bought stocks with mean-variance strategy in the very first day in July and sold in late September, his portfolio would loss more than a portfolio with tail dependence strategy. And also, if that investor did the same thing, which invested in a mean variance portfolio, in interval of October until December, that portfolio would also suffer more than a portfolio with tail dependence constraint. In conclusion, if market index return had bad return, the portfolio with tail dependence could have been outperform than mean variance portfolio.

However, the result of this paper is on assumption of stock and market return and the couple is approximately Student's t and t-copula. In practical, stock and market return and dependency model might not be the assumption similar to this paper. The portfolio allocation might be different, if the return and model is changed. Future studies should aim to allocate portfolio in a different assumption. For example, the return is still Student's t but the dependency would be Archemedean or Extreme value or closured property distribution. Lastly, in case of market index is predicted with good return, the tail dependence strategy might have adjusted to get a high return condition on the market is good.

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