PERFORMANCE COMPARISON BETWEEN LQR AND ANFIS CONTROLLER OF AN INVERTED PENDULUM SYSTEM

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Abstract— Inverted pendulum is a nonlinear and unstable dynamic system, thus providing a challenge to control engineers to stabilize it. This paper investigates the performance of few different control approaches that consist of modern and intelligent controllers. It involves the derivation of the mathematical modeling that includes the linearization of the model in order to be used with the linear controller. The works followed with designing those controllers and simulating it in MATLAB. After analyzing each controller performance, it can be observed that control strategies meet the design criteria. Overall, LQR controller has the best response, whereas fuzzy controller based on fusion function gives more flexibility in control action.

Index Terms—Fusion function, Fuzzy logic controller, LQR controller, Modeling

I. INTRODUCTION

The inverted pendulum system is a simple mechanical system which is a convenient model for the attitude control variety of systems such as space booster rocket and a satellite, automatic aircraft landing system, two way robot arms etc. As shown in Fig. 1 it consists of a cart, driven by a force that can move along a horizontal track, and a pendulum attached to the cart which can rotate freely in the vertical plane parallel to the track.

The inverted pendulum system is a highly nonlinear and unstable system, thus representing a challenging control problem. When the system is simulated the pendulum falls over quickly, so it must be actively balanced in order to remain upright. The characteristics of the inverted pendulum make identification and control more challenging [1].

The aim of inverted pendulum system is to drive the pendulum to its upright position and remains it there as well as maintaining the position of the cart. The control problem of stabilizing the system can be approximated by linearising the model, thus the linear feedback control such as PID control and pole placement control can be applied. In general, the control of this system by classical methods is a difficult task [2]

As for intelligent controller, multi-variables systems using fuzzy control theory have to face "rule explosion" problem [3],[4]. The method of state variables fusion is used to build a new controller in order to reduce the number of input variable. In this way, the problem of "rules Explosion" is solved by decreasing the number of fuzzy rules for the controller [5],[6].

This paper will describe designing a few types of controllers for inverted pendulum system that consist of modern controller and intelligent controller. State space pole placement controller represent as a modern controller, fuzzy logic based on fusion function represents the intelligent controller. An analysis of the performance will be carried out to the entire controllers, so that the best performance can be identified.

II. MATHEMATICAL MODELING OF AN INVERTED PENDULUM SYSTEM

As illustrated in Fig. 1, the physical system consists of a cart and a pendulum attached to the cart. The cart can move along a horizontal track (e.g. in the x direction), and the pendulum hinged by an angle θ from the vertical axis is able to rotate freely in the vertical plane parallel to the track.

The dynamical equations for the inverted pendulum system will be derived using "Lagrange Equations". The Lagrange equations use kinetic and potential energies of the system to determine the dynamical equations of the inverted system. Table 1 shows the parameters considered for the inverted pendulum system.

To derive the suitable mathematical model for an inverted pendulum system the Euler Lagrange method is used due to significant mathematical simplifications. The Lagrangian from (1) uses the difference between kinetic and potential energies of the system and is used with Euler Lagrange equation as shown in (2).

Table 1: Properties of the Inverted Pendulum System

Symbol	Description	Value	Unit
M	mass of cart	0.5	kg
m	mass of pendulum	0.2	kg
b	friction of cart	0.1	N/ms
L	length to pendulum center of mass	0.3	m
I	inertia of the pendulum	0.006	kg m^2
g	gravitational acceleration	9.81	m/s^2

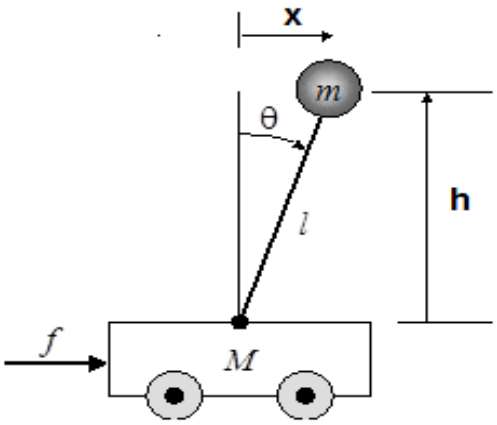


Fig. 1: Inverted pendulum system

$$L = T - U \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}} = Q_i \quad i = 1,2,\cdots r \tag{2}$$

The cart pendulum system has two degrees of freedom and can be fully represented using two generalized coordinates which are chosen as the horizontal displacement of the cart, x, and the rotational displacement of the pendulum, θ [7]. The kinetic energy of the system is the sum of the kinetic energy of each of the masses. The kinetic energy function for the cart mass is simply illustrated in (3).

$$T_{cart} = \frac{1}{2}M\dot{x}^2 \tag{3}$$

For the pendulum, the energy function can be derived as (6), from (4) and (5).

$$T_{pend.} = \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2 \tag{4}$$

where I is the moment of inertia around the pendulum's center of mass and $v_c$is the velocity of the pendulum's center of mass. Fig. 2 shows the velocity diagram of the pendulum.

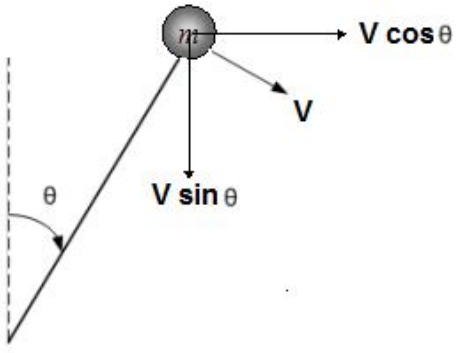


Figure 2: Velocity analysis of the pendulum

The center of mass coordinate is (x + lsinθ, lcosθ), therefore the v_c will be obtained as (5):

$$\begin{aligned} v_C &= v_x + v_y \\ v_x &= \dot{x} + l\dot{\theta}\cos\theta, \quad v_y = -l\dot{\theta}\sin\theta \\ v_C^2 &= v_x^2 + v_y^2 = \dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2 \end{aligned} \tag{5}$$

The angular velocity of the pendulum is simply $\omega = \dot{\theta}$. By substituting (4) and (5) into kinetic energy of the pendulum, the total kinetic energy is derived as (7).

$$T_{pend} = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 \tag{6}$$

$$T = T_{cart} + T_{pend} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 \tag{7}$$

Since the cart moves only in the horizontal direction, the potential energy of the system is determined entirely by the angle of the pendulum, which is:

$$U = mgh = -mgl\cos\theta \tag{8}$$

From Lagrangian function given in (1), the Lagrangian can be written as

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta \tag{9}$$

The generalized coordination of the system are $x$and θ, so the Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \tag{10}$$

By simplifying and rearranging (9) into (10), the system state equations are

$$\begin{cases} (M+m)\ddot{x} + b\dot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f(t) \\ ml\ddot{x}\cos\theta + (ml^2 + I)\ddot{\theta} + mgl\sin\theta = 0 \end{cases} \tag{11}$$

These dynamical equations must now be placed into state-space form. The state may be defined as $x=\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T$, the input as $u(t) = f$.

In order to get a linear model, linearization process must be done to the nonlinear dynamic equations of the inverted pendulum system. The model can be linearized if the following conditions;the equilibrium point of the system, assuming that the pendulum is up, is (θ=π, x=0).

$$\begin{aligned} \cos\theta &\approx \cos\pi + (\pi-\theta)(-\sin\pi) = -1 \\ \sin\theta &\approx \sin(\pi) + (\pi-\theta)\cos(\pi) = \pi - \theta \approx -\theta \end{aligned} \tag{12}$$

By simplifying the nonlinear terms as given by in (12), and substituting them into the dynamic equations in (11) the linearized differential equations of the system obtained as

$$\begin{cases} (M+m)\ddot{x} + b\dot{x} - ml\ddot{\theta} = f(t) \\ -ml\ddot{x} + (ml^2 + I)\ddot{\theta} - mgl\theta = 0 \end{cases} \tag{13}$$

For linear state control of the inverted pendulum, it is necessary to convert the dynamic equations in (13) to state-space equation in matrix form. Cart's position and pendulum's angle are outputs of the system. By substituting all the parameter values from Table 1, the state-space and output representations of the system will be as in (14).

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \tag{14}$$

CONTROLLER DESIGN

The controller simulation is implemented via two control strategies, namely linear quadratic regulation (LQR) controller, fuzzy logic controller (FLC) based on fusion function. The control problem is to design a controller which computes the applied force (F) to cart in such a way that the pendulum reaches its upright position and remains there, as well as maintaining the initial position of the cart on the horizontal plane.

For decision making of controller design and examining the performance of it, a few design requirements have been set. In this design, we consider three considerations to be met which are settling time (T_s) less than 5 second, percentage of overshoot (%OS) less than 22.5%, as well as steady-state error is within 2%. The five second is chosen to determine the effectiveness of the designed controllers in term of fast response. Whereas, the limitation of overshoot are required to avoid the pendulum reach the horizontal position.

A. LQR Controller Design

Full state feedback control, observer and optimal control are some types of state space controller techniques. This paper analyzes a linear quadric regulation (LQR) approach, and the schematic of the full-state feedback system is shown in Fig. 3. From this figure, Rref is desired position, y is output position, and K is full-state feedback gain.

For the LQR controller design, state equation and output equation as in (14) will be used. The characteristic equation for the closed-loop system is given by the determinant of *[sI(A-BK)]*, where *I* is the identity matrix, while *A* and *B* are the system matrix and input matrix respectively from state space equation. For this system, there are four state variables, hence the *A* and *B*K* are both 4×4 matrices. There should be four poles for this system.

The *'lqr'* function (in Matlab) allows choosing two parameters, regulator (*R*) and quadratic (*Q*), which will balance the relative importance of the input and state in the cost function that are trying to be optimized. The simplest case is to assume *R*=1, and $Q=C^T*C$, where *C* is output matrix from (14) and C^T is matrix transpose of *C*. The controller can be tuned by changing the nonzero elements-x and y- in the *Q* matrix to get a desirable response.Thus, those elements will be used to weight the output response. The strategy is described in Matlab command as shown in (15).

```
R = 1;
Q =[x 0 0 0; 0 0 0 0; 0 0 y 0; 0 0 0 0];
K = lqr (A, B, Q, R)
```
(15)

From above command, by increasing x and y, the settling time (T_s) and rise time (T_r) can be decreased. For this design, the value of x is set to 5000 as well as y to 100 and let *R* remain one. The following values for state feedback matrix were found:

K = [-70.7107 -37.8345 105.5298 20.9238]

By using State Feedback method only, there would be a very large steady state error. Therefore, in order to eliminate the steady-state error we have to optimize the design by adding a constant gain *Nbar* after the reference input as shown in Fig. 4 [5]. The value *Nbar*can be determined by using the user defined function *'rscale'*.

N = rscale (A, B, C, D, K) (16)

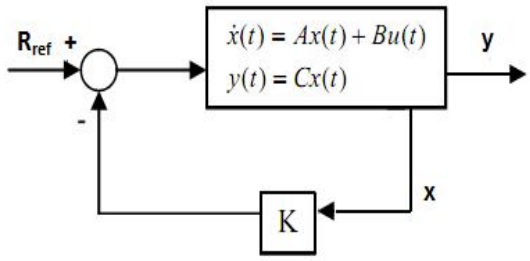


Fig. 3: State feedback control configuration

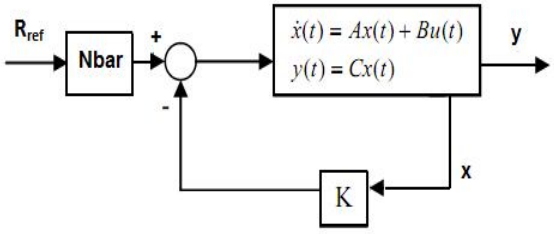


Fig. 4: State feedback control configuration with the input gain, *Nbar*

In this case matrix D is equal to zero. Finally we will get the new matrix equation for *A, B, C* and *D* through this controller which are A_c=*A*–*B***K*,B_c=*B***Nbar*,C_c=*C* and D_c=*D*.

B. State Variables Fusion Based ANFIS Controller Design

B.i State Variables Fusion

The inverted pendulum has 4 state variables:$x, \dot{x}, \theta, \dot{\theta}$, if the normal fuzzy control method is used, the number of rules is equal to 7^2. This can cause "rule explosion" [7]. In this section to solve this problem, optimal control theory and fuzzy control strategy are combined. We use fusion technology to design a linear fusion function, transform multiple variables into a comprehensive error E and the rate of change of error EC, which greatly simplified the ANFIS controller [5],[8].

The linear fusion function is constructed in the basis of LQR control. For fusion function design steps are given as follow [6]:

1. The state feedback matrix K was calculated in section (A) through LQR theory, which K equals [-70.7107 -37.8345 105.5298 20.9238].
2. The description of fusion function $F_1(x)$ using matrix K is as (17):

$$F_1(X) = \begin{pmatrix} \frac{K_x}{|K|_2} & \frac{K_\theta}{|K|_2} & 0 & 0 \\ 0 & 0 & \frac{K_{\dot{x}}}{|K|_2} & \frac{K_{\dot{\theta}}}{|K|_2} \end{pmatrix} \tag{17}$$

where

$$|K|_2 = \sqrt{K_x^2 + K_\theta^2 + K_{\dot{x}}^2 + K_{\dot{\theta}}^2} \tag{18}$$

The fusion function based on state feedback matrix K is calculated as:

$$F_1(X) = \begin{pmatrix} -0.67005 & 1 & 0 & 0 \\ 0 & 0 & -1.8082 & 1 \end{pmatrix} \tag{19}$$

3. Reduce the dimension of input variable, by calculating comprehensive error E and the rate of change of error EC as defined in (20).

$$\begin{pmatrix} E \\ EC \end{pmatrix} = F_1(X) X^T \tag{20}$$

In this case,

$$E = K_x.x + K_\theta.\theta$$
$$EC = K_{\dot{x}}.\dot{x} + K_{\dot{\theta}}.\dot{\theta} \tag{21}$$

C.ii AdaptiveNeural Fuzzy Controller Design

The architecture and learning rules of adaptive neural fuzzy inference system (ANFIS) is described here. In ANFIS, Takagi-Sugeno type fuzzy inference is used. An ANFIS architecture with two inputs x and y, and one output z is shown in Fig.5. The architecture has five layers. The nodes in the same layer have similar functions as described below:

In layer 1, output of this layer is the membership grade of a fuzzy set and it specifies the degree to which the given input x satisfies the quantifier A_i. In layer 2, every node's output is the product of all the incoming signals. These nodes perform the fuzzy AND operation. Each node output represents the firing strength of a fuzzy rule. In layer 3, every node calculates the ratio of the rule's firing strength to the sum of all rules' firing strengths:

$$O_{3,i} = \overline{wi} = \frac{w_i}{w_1 + w_2}, \quad i = 1,2 \tag{22}$$

wherew_i is the firing strength of a rule. The outputs of this layer are called normalized firing strengths. In layer 4, every node *i* is a square (adaptive) node with a node function

$$O_{4,i} = \overline{w_i} f_i = \overline{w_i}(p_i x + q_i y + r_i) \tag{23}$$

where$\overline{w_i}$ is the normalized firing strength from layer 3, and $\{p_i, q_i, r_i\}$ is the parameter set of this node. Parameters in this layer are referred to as consequent parameters. The single node in layer 5 is a fixed node labeled as Σ, whichcomputes the overall output as the summation of all incoming signals as where $O_{5,i}$ denotes the output of the i^{th} node in layer 5.

$$O_{5,i} = \sum_i \overline{w_i} f_i = \frac{\sum_i w_i f_i}{\sum_i w_i} \quad (24)$$

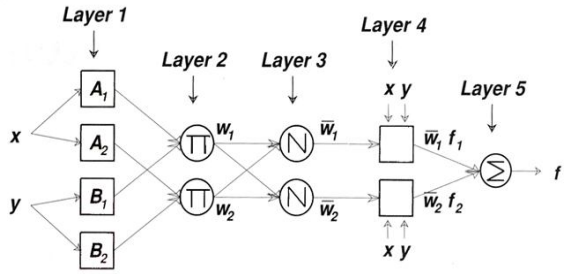


Fig.5: ANFIS architecture

Table 2: Two passes in hybrid learning algorithm

	Forward pass	**Backward pass**
Premise parameters	Fixed	Gradient descent
Consequent parameters	Least-square estimator	Fixed
Signals	Node outputs	Error signals

Learning algorithm in ANFIS is called hybrid learning algorithm, which is the combination of gradient descent and least squares methods. Table 2 explains the hybrid learning algorithm.

In the forward pass, the node outputs go until layer 4, and the consequent parameters are identified by the least square estimate. While in backward pass, the error signals propagate backward and the premise parameters are updated by the gradient descent method.

The implementation of ANFIS on inverted pendulum system is started by making a training data which can describe the parametric values of the system.

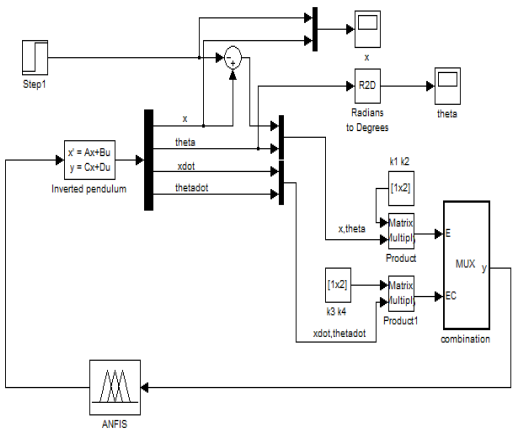


Fig.6: Simulation model of inverted pendulum based on ANFIS

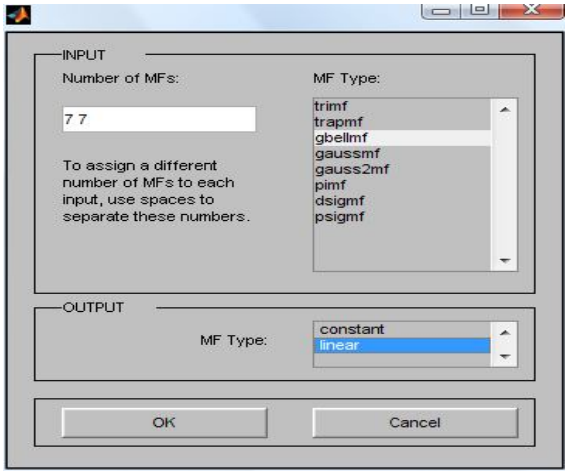


Fig. 7: Initialization of membership functions

Training vector has two input parameters that of comprehensive error E and the rate of change of error EC, as defined in (21), also with the output as force of fusion function with LQR controller which derives the cart. Checking data is created and the data are loaded in the ANFIS editor GUI. In this simulation, the LQR controller is applied to the inverted pendulum system to get input and output data. The sampling time is set to 0.01 sec. 1000 sample data is selected as the ANFIS controller training data. The simulink structure of inverted pendulum controller based on adaptive neural network simulation is shown in Fig.6.

After acquisition of training data, next step is establishment of ANFIS network structure. For ANFIS model, first step is to determine the output and input variables membership functions. Bell-shaped membership function of the prototype is the probability distribution function and can better reflect reality. The system uses bell shaped membership function to represent the first i input membership function. This step is shown in Fig. 7.

In next step, for adjusting the parameters of ANFIS fuzzy model the hybrid learning algorithm is chosen. ANFIS learning methods integrated use of gradient descent learning method and least square method (LSE).

After selecting initial membership function shown in Fig. 7, Sugeno fuzzy model identification methods and hybrid learning algorithm are selected. By using the grid partition method to obtain the after training method of the system, fuzzy rules and modified membership function are identified. Closing value of the error (Error Tolerance) is set to 0 and the training times (epoch) is set to 30. After training the FIS structure, the modified membership functions are shown in Fig. 8 and Fig. 9 for E and EC respectively. The value span of E is [-105 105], while EC is [-35 35].

In final step, the FIS output can be exported to workspace for the ANFIS controller to make its rules The ANFIS controller is attached to the plant and simulated.

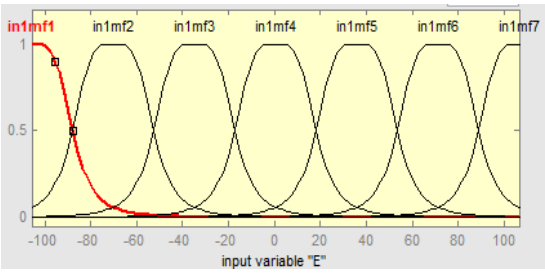


Fig. 8:The membership functions of variable E

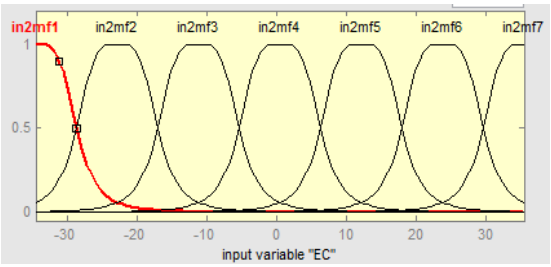


Fig. 9:The membership functions of variable EC

RESULTS AND ANALYSIS

The graph in Fig. 10 shows the overall step responses of the cart's position for controllers. The designed LQR controller has an overall better performance than ANFIS controller; the LQR has the rising time (T_r) of 0.41 second while the ANFIS controller has the 2.4 seconds. Moreover, LQR settles very fast compare to ANFIS which is 2.04 seconds, whereas ANFIS has the settling time of 3.68 seconds. For the overshoot percentage (OS %), LQR does not have overshoot while the ANFIS controller has the overshoot percentage of 1.5%. As a whole, all the characteristics have fulfilled the design criteria for both controllers.

The graph in Fig. 11 shows the overall output responses of the pendulum's angle between LQR, and ANFIS based on fusion function controllers. As can be seen, all the responses reached zero which stabilize the pendulum vertically. Based on the results, LQR has the fastest settling time (T_s) of 3.34 seconds, followed by ANFIS with the value of 4.34 seconds. Unfortunately, both controllers have exceeded the maximum range limitation of 20 degrees; however ANFIS controller has the better response whereas LQR performs the worst. For stabilizing the pendulum's angle ANFIS controller based on fusion function has better performance which satisfies the design criteria very much.

Both results had shown that to control cart's position, LQR controller is the best; whereas to control the pendulum's angle Adaptive neural network fuzzy controller shows better performance. In addition, by using fusion function the dimension of the input varieties are reduced and 'rule explosion' problem is solved. The results show that control objective in stabilizing the inverted pendulum has been achieved well; it not only has quite good stability, but also is able to make the cart of the pendulum moving to the desired place.

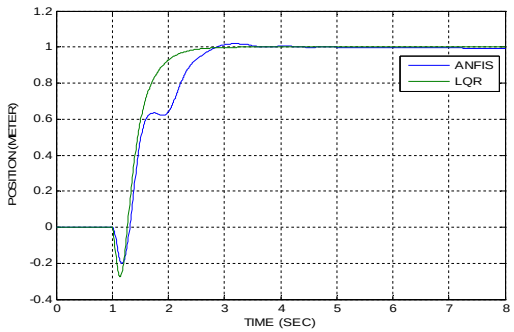


Fig. 10: Output response of cart's position

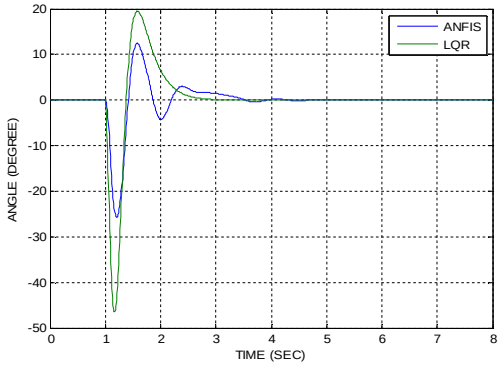


Fig. 11: Output response of pendulum's angle

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