GRASP AND STATISTICAL BOUNDS FOR HEURISTIC SOLUTIONS TO COMBINATORIAL PROBLEMS

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Abstract - The quality of a heuristic solution to a NP-hard combinatorial problem is hard to assess. A few studies have advocated and tested statistical bounds as a method for assessment. These studies indicate that statistical bounds are superior to the more widely known and used deterministic bounds. However, the previous studies have been limited to a few heuristics and combinatorial problems and, hence, the general performance of statistical bounds in combinatorial optimization remains an open question. This work complements the existing literature on statistical bounds by testing them on the metaheuristic Greedy Randomized Adaptive Search Procedures (GRASP) and four combinatorial problems. Our findings confirm previous results that statistical bounds are reliable for the p-median problem, while we note that they also seem reliable for the set covering problem. For the quadratic assignment problem, the statistical bounds have previously been found reliable when obtained from the Genetic algorithm whereas in this work they have been found less reliable. Finally, we provide statistical bounds to four 2-path network design problem instances for which the optimum is currently unknown.

Key Words - Combinatorial Problems, GRASP, Statistical Bounds, Statistical Optimum Estimation Techniques

I. INTRODUCTION

Solutions to combinatorial problems have been extensively studied in the field of discrete optimization. The applications to combinatorial problems are found on classical problems like traveling salesman problem, location problem, network design and scheduling problems. For a simple problem, the complete algorithm guarantees to find the optimal solution. However, for less simple problems, complete algorithms will fail to find the optimal solution in the polynomial time, in which case the problem is known as NP-hard and therefore computationally intractable.

Many algorithms for NP-hard problems have been developed in the last 30 years. Metaheuristics is a high-level strategy for exploring search spaces by using different methods [1]. Examples are GRASP, Tabu Search, Iterated Local Search, Variable Neighborhood Search, Genetic search and Simulated Annealing. In contrast to the complete algorithms, metaheuristics try to locally search for good solutions, thereby significantly reducing the combinations to evaluate.

They usually start from some initial solution and iteratively try to replace the current solution by a better solution in an appropriately defined neighborhood of the current solution [1]. In doing so, the guarantee of finding the optimal solutions to the problem has been sacrificed.

The prevailing practice is to run the heuristic for a pre-specified number of iterations or until improvement in the solution becomes infrequent. However, for a specific problem instance, such practice does not readily lend itself to the assessment of the quality of the solution [2]. One approach to assess the quality is to seek the deterministic bounds [3], [4]. This approach is however applicable to a few heuristics and sensitive to parameter settings. For an unsuitable parameter setting, the gap of the bounds is usually large and thus uninformative. Another approach is to use the statistical property of the solution. Specifically, the statistical approach implies point estimation to the optimum with an associated uncertainty interval at some confidence level. Golden and Alt [5] did pioneering work on statistical bounds followed by others in 1980s. Derigs [6] conducted computer experiments with regard to the confidence limits for travelling salesmen and quadratic assignment problem instances by applying k-interchange heuristics. Although Derigs' conclusions [6] were generally favorable for the statistical bounds, his work received little attention during three decades in the practice of combinatorial optimization. A recent work [7] reviewed four statistical optimum estimation techniques that provides a comprehensive framework of optimum estimation. From their concluding discussion, it is clear that for statistical optimum estimation techniques to be well understood and practically useful in combinatorial optimization a great number of varying problem instance specific computer experiments are warranted. In addition to computer experiments in [6], Carling and Meng[2], [8] studied the statistical confidence for the p-median problems using simulated annealing and vertex substitution algorithms, while [9] tested statistical bounds using the Genetic algorithm on QAP. These studies suggest that statistical bounds do provide a good assessment of the quality in the heuristic solution with a greater precision than deterministic bounds. In this paper, our aim is to complement the existing knowledge about the performance of statistical bounds by conducting novel computer

experiments of one metaheuristic algorithm, GRASP, on four combinatorial problems. The work is inspired by a recent study [10]that developed a probabilistic stopping rule for the GRASP heuristics. They studied the power of the statistical distribution and gave the likelihood of a better solution for a certain, additional number of iterations. This is informative if a pre-set parameter, β, is set since the expected number of better solutions in the next run can be estimated. Based on 1 million pre-run iterations, experiments are interrupted as soon as the probability of obtaining better solutions in the next 1 million iterations is lower than the threshold parameter β. However, they found the approach to fail when $\beta \leq 10^{-4}$ for QAP and set covering problem instances. Considering the optimal value is usually over 10 times of the standard deviation away from the mean, the threshold may not be sufficiently small. Further, the method in [10] also relies on the assumption that distribution is continuous when calculating beta, while the solutions follow a discrete distribution. Since the gap between the statistical bounds can be used as a stopping criterion, this work is also complementing the work on stopping rules for GRASP in [10].

The remaining of the paper is organized as follows. Section II gives a short description of the methodology; Section III introduces GRASP and the problem instances; Section IV gives the results; and the final section provides concluding remarks.

II. METHODOLOGY

The statistical optimum estimation techniques (SOETs) are extensively reviewed [7]. For future (computer) studies of SOET they proposed an evaluation framework with associated notation. A problem instance I is a particular combinatorial optimization problem with fixed constraint and objective function coefficients [9]. A problem class $\mathcal{I}$ is a group of problem instances where $I \in \mathcal{I}$. Similarly, $\mathcal{H}$ is the collection of all possible heuristics. For a particular heuristic algorithm $H \in \mathcal{H}$, the number of replicates, n, is usually arbitrarily determined by the experience. SOETs is a set consisting of the complete combination of problems and algorithms: $\mathcal{I} \times \mathcal{H}^n$. Thus, the task of summarizing all possible $\mathcal{I} \times \mathcal{H}^n$ is not straightforward, because same heuristics could perform completely differently on different problems. For this reason, it may be necessary to give customized statistical bound for each element in $\mathcal{I} \times \mathcal{H}^n$. They also noted that studies on statistical bounds for combinatorial problems are rare. This far, Derigs [6] found that statistical bounds are more useful than deterministic bounds for harder problems and that the k-interchange heuristics he studied gave more competitive statistical bounds for QAP than TSP. Derigs [6] also pointed out that the Weibull hypothesis derived from extreme-value theory is a proper approach. Carling and Meng [2] examined the Weibull (and the Gumbel) point estimator of the optimum as well as the first and second order Jackknifing (JK) point estimator on 40 p-median problem instances using Simulated Annealing as metaheuristic. Moreover, they proposed a SR statistic (hereafter referred to as SR and see Section IV for detailed definition) to determine if statistical bounds could be deemed reliable (and cautioned for the use of statistical bounds unless SR is smaller than 4). They also found the second order JK estimator preferable for both point and interval estimation. Further, Carling and Meng[8] found the statistical bounds obtained from Simulated Annealing and Vertex Substitution to be reliable and much more efficient than deterministic bounds derived from Lagrangian relaxation for the same 40 p-median problem instances as above. Furthermore, they claimed that statistical bounds may be computed based on a small number of replicates ($n = 10$). Meng[9] showed that the 4th order JK-estimator is an attractive SOET for difficult QAP instances when using the Genetic algorithm. Meng[9] also compared statistical bounds to several methods to derive deterministic bounds and concluded that in general statistical bounds were competitive to the best deterministic ones.

We follow the notation in [2], but generalize them from the p-median to all problem instances originally studied in [10]. They are p-median problem (PMP), quadratic assignment problem (QAP), set k-covering problem (SCP) and 2-path network design problem (2-PNDP). We introduce them in details in Section III. The following notation applies throughout the paper:

z_p: a feasible solution of a specific instance (I);
A: the set of all feasible solutions to I, $A = \{z_1, z_2, \dots, z_N\}$;
$g(z_p)$: the value of the objective function at z_p;
θ: the minimum of $g(z_p)$, $\theta = \min_A g(z_p)$;
$\hat{\theta}$: an estimator of θ;
n: the number of replicates of the metaheuristic with unique random starting values;
$\tilde{x}_i$: the heuristic solution of the ith replicate, $i = 1, 2, \dots, n$;
$\tilde{x}_{(i)}$: the ith order statistic of n heuristic solutions.

Even though the problem instances are fixed, the heuristics generate random solutions. Thus, both $\tilde{x}_i$ and $\tilde{x}_{(i)}$, $i = 1, 2, \dots, n$, are considered as random variables. A random variable, X, possesses one unique type of distribution that characterizes how the possible values for X distribute providing a relative likelihood. There are two types of distributions: discrete distribution and continuous distribution. We focus on the later one in this study. For the continuous distributions, only the values within an interval are non-zero. Thus, one interval corresponds

to one probability that indicates how likely the values within the interval can be taken if we draw a single observation from X. The ending points of the interval are known as the lower bound and the upper bound. Since $\hat{\theta}$ is a function of $\tilde{x}_{(i)}$, $\hat{\theta}$ is also a random variable. Thereafter, the desired statistical bound in this study is formulated in terms of probability, $P(\hat{\theta}_L \leq \hat{\theta} \leq \hat{\theta}_U) = 1 - \alpha$, where $\hat{\theta}_L$ is the lower bound, $\hat{\theta}_U$ is the upper bound and α the confidence level. Similarly, the one-side statistical bound is $P(\hat{\theta}_L \leq \hat{\theta}) = 1 - \alpha$. To align with the work of [2], [8], [9] and [10], we will examine the Weibull point estimator, $\hat{\theta} = \hat{\theta}_W$, and the second order JK point estimator, $\hat{\theta} = \hat{\theta}_{JK}^{(2)}$. The Weibull point estimator is one of the application of the extreme-value-theory approaches [7], where $\hat{\theta}_W = \tilde{x}_{(1)}$ is the statistical upper bound. The statistical lower bound $\hat{\theta}_L$ is $\hat{\theta}_W - \hat{c}$ with the confidence level of $\alpha = e^{-n}$, where $\hat{c}$ is the estimated shape parameter of Weibull distribution [11]. One goodchoice is $\hat{c} = \tilde{x}_{[0.63(n+1)]} - (\tilde{x}_{(1)}\tilde{x}_{(n)} - \tilde{x}_{(2)}^2)/(\tilde{x}_{(1)} + \tilde{x}_{(n)} - 2\tilde{x}_{(2)})$ as suggested by [6].

The sample minimum is a natural choice for estimating the lower bound as the truncation point. Considering that the sample minimum is a biased estimator, a batch of extensions of the estimators have been developed. The second order JK point estimator is a direct application of the truncation-point approaches [7], where $\hat{\theta}_{JK}^{(2)} = 3\tilde{x}_{(1)} - 3\tilde{x}_{(2)} + \tilde{x}_{(3)}$, whereas the upper statistical bound is $\tilde{x}_{(1)}$. The lower statistical bound $\hat{\theta}_L$ is obtained by $\hat{\theta}_{JK}^{(2)} - 3\sigma^*(\hat{\theta}_{JK}^{(2)})$ where $\sigma^*(\hat{\theta}_{JK}^{(2)})$ is the standard deviation of $\hat{\theta}_{JK}^{(2)}$ obtained from bootstrapping the n heuristic solutions. Carling and Meng [2] proposed using 1,000 bootstrap samples such that the confidence level is nominally 99.9% ($\alpha = 0.001$) based on the assumption that the JK estimator follows a Normal distribution.

III. GRASP AND THE PROBLEM INSTANCES

We give a template for the GRASP and the test problem instances. GRASP is an iterative randomized sampling metaheuristic and was first introduced by [12]. Three test problems: PMP, QAP and SCP are used to evaluate the statistical bounds since the optimal solutions are available in the public libraries [13] and [14]. The fourth problem 2-PNDP and its four instances is a random data set provided to us by [10]. The optimal solutions to these four problem instances are unknown. Statistical bounds are discussed in Section IV.GRASP is an iterative algorithm with random starting values. The pseudo-code is shown in Fig. 1. Line 1 inputs a specific problem instance by identifying the feasible solutions z_p and the object function $g(z_p)$. Line 2-6 run the construction stage and the local search stage when the stopping criteria is not reached. The number of iterations is pre-set, but can be completely different for each problem. Line 7 returns the best solutions.

A. Description of GRASP

The construction and local search stages are problem-dependent and should be customized for specific problem. In the construction stage, the probabilistic component is characterized by randomly selecting one of the best candidates in the list, restricted candidate list (RCL), but not necessary the top candidate ([12]). RCL is formed by a number of the elements which add smallest incremental cost to the current solution. RCL is updated each time as one element is selected until a feasible solution is formed. The output from the construction acts as the input of the local search stage.

```
GRASP
  1  Instance Input;
  2  While (stopping criteria not reached) {
  3     Construction stage(RCL) → Solution 1;
  4     Local search stage(Solution 1) → Solution 2;
  5  Update Solution 2 → Better Solution;
  6  Check stopping criteria};
  7  Return Best Solution;
End
```

Fig. 1 Pseudo-code of GRASP

The local search stage tries to improve the solution by searching in neighbor solutions. The definition of neighbor is flexible and also problem-dependent. The local optimum solution is found when the local search stopping criteria is reached.

Like other heuristics, the value of applying GRASP on combinatorial problems is to avoid being trapped in local minima. A specific characteristic of GRASP is that the parallel starting procedure is able to search the solutions in a wide range, though new iterations will not learn from previous iterations.

B. Problems and instances

Four combinatorial problems are to be described and studied. We follow [10] and examine the same four instances for each of the four problems.

1. p-median problem

The discrete p-median model was first introduced by [21] andwas shown as NP-hard problem in [22]. The goal with the model is to find p supply nodes which minimize the summed distances between each demand node and their nearest supply node. This problem can be formulated to minimize

$$f_1 = \sum_{i=1}^{q}\sum_{j=1}^{q} w_i d_{ij} x_{ij}.$$

subject to $\sum_{j=1}^{q} x_{ij} = 1$ and $\sum_{j=1}^{q} x_{jj} = p$, where f is the value of objective function. q is the number of demand locations and also stipulated to be the number of candidate nodes. $w_i$is the weight of each demand location. d_{ij} is the distance from demand location i to the center j. $x_{ij}$is a binary variable: taking on 1 if location i is allocated to center$j$. Test instances are shown in Table I ([13]). The number of candidate nodes are between 200 and 800 and the number of supply nodes are between 67 and 200. In the construction stage of GRASP, one of the nodes in RCL is selected and added into the solution at a time until all p nodes are included. In the local search stage, nodes in the neighbors are examined. As indicated by [23], substitution occurs when a low-cost neighbor is found.

1) Quadratic assignment problem

QAP was first defined by [24] and it is NP-hard ([25]). Consider two square matrixes $A = (a_{ij})_{N_Q}$and $B = (b_{ij})_{N_Q}$. A set Y of all permutations of $\{1,2,\dots,N_Q\}$ is formed by $N_Q!$ elements and $y \in Y$. The objective function of QAP is to minimize

$$f_2 = \sum_{i=1}^{N_Q}\sum_{j=1}^{N_Q} a_{ij}b_{y(i)y(j)}.$$

Four problem instances are given in Table II including the instance name and size ([14]). When $N_Q > 15$, the QAP becomes very difficult to solve. The computational experiments with GRASP is similar to the PMP. Random initial solutions in RCL are selected at the construction stage. Interchanges take place between every two possible elements in B (or A) for the local search.

2) Set k-covering problem

SCP is a NP-complete problem ([15], [16])of covering the rows of an m_s-row, n_s-column zero-one matrix (S_{ij}) by a subset of columns at minimum cost. Suppose $x_j = 1$ if column j is in the solution with cost c_j, otherwise $x_j = 0$. The object is to minimize

$$f_3 = \sum_{j=1}^{n_s} c_j x_j.$$

where $\sum_{j=1}^{n_s} s_{ij}x_j \geq k$ guarantees that each row is covered at least k times. The test instances are described in Table III ([13]). Implementation for the GRASP is illustrated in [17].

3) 2-path network design problem

The fourth problem does not have a known optimum to the four instances, which means that the statistical bounds cannot be checked with the optimum. For these problem instances we will provide statistical bounds as a complement to the few studies that have tried to provide a solution to the problem instances of 2-PNDP in [10] and [18].

Given a nonnegative edge supply graph (E, V) and a subset of $|V|$ pairs of nodes (demand graph), the 2-PNDP is to find the minimum weighted edges in the supply graph such that each

TableITest instances of the p-median problem

instances	q	number of edges	p
pmed10	200	800	67
pmed15	300	1800	100
pmed25	500	5000	167
pmed30	800	7200	200

Table II Test instances of the quadratic assignment problem

instances	N_Q
tai30a	30
tai35a	35
tai40a	40
tai50a	50

Table III Test instances of the set k-covering problem

instances	m_s	n_s	k
scp42	200	1000	2
scp47	200	1000	2
scp55	200	2000	2
scpa2	300	3000	2

Table IV Test instances of the 2-path network design problem

instances	|V|	|E|	K
2pndp50	50	1225	500
2pndp70	70	2415	700
2pndp90	90	4005	900
2pndp200	200	19900	2000

pair in the demand graph is connected by only one or two edges, i.e. minimizing

$$f_4 = \sum_{e\in E}^{n} c_e x_e.$$

where c_e is the weight of the selected edge and $x_e \in \{0,1\}$ indicates if the edge is selected from E. Four test instances are shown in Table IV. The application of GRASP on 2-PNDP is described in [19] and [20]. In the construction stage, a random order of the demand pairs is constructed. Starting from the first pair, one of the edges in RCL with the lowest weights is added to the solution set. This procedure continues one after another pair in the demand graph until all demand pairs have been added. In the local search stage, a random order of demand pairs is generated. Each demand pair is picked from the beginning to the end and paths with lower costs are searched for each pair. As soon as the last pair has been reached, a new search from the

beginning starts. The local search stops when no improvement can be found.

IV. RESULTS

In this section, we present the main findings in estimating the optimal solutions. We have run $n = 100$ replicates for each problem instance for the same reason as [2] and [8]. Considering that the neighbors for local search are highly problem-dependent and the computing efforts for the number of the local searching stage increases as the problem complexity increases, the number of local search, $\mathcal{L}$, varies by problem instance. For example, we take $\mathcal{L} = 20{,}000$ for pmed10 and $\mathcal{L} = 150{,}000$ for pmed30 while $\mathcal{L}$ varies between 1 million and 1.5 million for the most difficult problem being QAP. There is no simple way to compare our setting of $\mathcal{L}$ on the instances to previous work on statistical bounds, but [8] provided information on their choice of iterations for Simulated Annealing and Vertex Substitution in terms of computing time. Our setting in the implementation of GRASP on the instances implies that the heuristic runs for equally long time as in experiments for PMP and similarly for the SCP [8]. The setting for QAP implies that the computing time is much larger. As pointed out in Section III, the local search stage for 2-PNDP varies in each replicate and is influenced by the solutions in the construction stage.

Table V Point estimators, relative bias (‰) and SR

instances	θ	$bias(\hat{\theta}_{JK}^{(2)})$	$bias(\hat{\theta}_W)$	SR
pmed10	1255	-1.59	0.80	7.10
pmed15	1729	6.36	6.36	5.70
pmed25	1828	10.30	12.58	6.90
pmed30	1989	3.02	4.02	2.24
tai30a	909073	25.72	22.34	9.39
tai35a	1211001	23.98	25.15	8.17
tai40a	1569685	25.59	28.99	7.97
tai50a	2469398	30.78	33.85	1.69
scp42	1205	25.73	26.56	9.94
scp47	1115	11.66	21.52	8.95
scp55	550	16.36	20.00	6.38
scpa2	560	41.07	39.29	3.98

The iterations stop when there is no improvement in the local search stage. Thus, from the empirical study of 2-PNDP, we let $\mathcal{L}$ be at most 10,000. One experiment is defined as running GRASP $n = 100$ times for each problem instance. After each experiment, we also evaluate the statistic SR:

$$SR = \frac{1000\sigma(\tilde{x}_i)}{\hat{\theta}_{JK}^{(2)}}.$$

SR measures the similarity amongst the solutions potentially allowing one to judge how far out in the tail and close to the optimum solutions they are. Hence, a large SR indicates the $\tilde{x}_i$:s are not near to the optimum. As noted above, $SR < 4$ is a sufficient condition for statistical bounds to be reliably applied for simulated annealing and vertex substitution heuristics on PMP ([2], [8]). For GRASP it proved difficult to lower SR to a desirable level of about 4 within a reasonable computing time, and for most instances SR was found in the range of 5 to 10. Next, we evaluate both the point estimator and the coverage rate, i.e. whether the statistical bounds contain the optimum or not. Table V shows the point estimators $\hat{\theta}_{JK}^{(2)}$ and $\hat{\theta}_W$. We show the relative bias in millesimal, i.e. $\text{bias}(\hat{\theta}) = 1000 \times \frac{\hat{\theta}-\theta}{\theta}$.

In Table V, θ is the known optimal value. Column 3-4 is the relative difference between the estimator and the optimal value. And the last column is SR. The results coincide with those in [10] in terms of the distance to the optimal solutions: PMP is the easiest problem and gives smallest bias among three problems. QAP and SCP are harder problems. And $\hat{\theta}_{JK}^{(2)}$ for PMP always provides lower point estimation than $\hat{\theta}_W$. For QAP $\hat{\theta}_{JK}^{(2)}$ outperforms in three most complex instances: tai30a, tai40a and tai50a and also so on the three simpler instances of SCP. It is unsurprising that the harder instances of the same problem had lower SR considering that substantially more local searches were imposed on them.

The statistical bounds further state the precision of the estimators. To examine this empirically, we take a sample of size 10 with replacement from $n = 100$ replicates of each instance. This means that we approximate the true distribution of solutions obtained from infinitely many replicates on an instance, by the empirical distribution obtained from 100 replicates in order to reduce the time required for an experiment and allocate this time for conducting more experiments. We do this 1000 times and compute the lower bounds for the two estimators and compute the proportion of times that the lower bound and the upper bound contains θ. This proportion is also known as the coverage. Further, the average interval is the mean value of 1000 lower bounds and upper bounds. In Table VI, coverage and average statistical bounds for each problem instance are given for both estimators. As shown in Table VI, $\hat{\theta}_{JK}^{(2)}$ always produces higher coverage but wider intervals than $\hat{\theta}_W$. This is straightforward because one cannot expect for both good intervals and good coverage. The cost of wider interval is modest for the test instances while the coverage improves substantially. Even though all coverages are greater than 0, QAP still performs worst in terms of coverage among those three problems. One reason is that GRASP tends to be trapped in the local optima and is difficult to escape. On the contrary, PMP and SCP are

easier to cover the optimal value than QAP even though SCP is classified as harder problem. This can be seen from the high coverage for PMP and relatively high coverage for SCP. In this sense, 1000 samples and $SR < 10$ are enough to cover the known optimum. Like PMP, 2-PNDP is also classified as simple problem in [10] because a threshold $\beta = 10^{-5}$ quantilelower tail valuesare obtained.

Table VI Statistical bounds

instances	Cov. $\hat{\theta}_{JK}^{(2)}$	Bounds $\hat{\theta}_{JK}^{(2)}$	Cov. $\hat{\theta}_W$	Bounds $\hat{\theta}_W$	SR
pmed10	1.00	[1231, 1262]	0.84	[1246, 1262]	7.10
pmed15	0.99	[1715, 1746]	0.40	[1729, 1746]	5.70
pmed25	0.95	[1808, 1864]	0.22	[1838, 1864]	6.90
pmed30	0.99	[1980, 2001]	0.28	[1992, 2001]	2.24
tai30a	0.69	[904K, 932K]	0.13	[918K, 932K]	9.39
tai35a	0.82	[1202K, 1234K]	0.13	[1218K, 1234K]	8.17
tai40a	0.45	[1569K, 1625K]	0.01	[1594K, 1625K]	7.97
tai50a	0.03	[2507K, 2567K]	0.02	[2536K, 2567K]	1.69
scp42	0.92	[1182, 1248]	0.26	[1212, 1248]	9.94
scp47	0.90	[1099, 1144]	0.27	[1122, 1144]	8.95
scp55	0.90	[543, 563]	0.21	[554, 563]	6.38
scpa2	0.45	[559, 588]	0.09	[576, 588]	3.98

Table VII Point estimator and statistical bounds for 2-pndp

Instances	$\hat{\theta}_{JK}^{(2)}$	bounds $\hat{\theta}_{JK}^{(2)}$	$\hat{\theta}_W$	bounds $\hat{\theta}_W$	SR
2pndp50	322	[313, 323]	323	[320, 323]	3.66
2pndp70	466	[461, 467]	467	[465, 467]	2.71
2pndp90	635	[626, 638]	638	[634, 638]	3.91
2pndp200	1381	[1374, 1384]	1384	[1380, 1384]	2.42

Thus, the true coverage is expected to be similar to the PMP. Since optimal solutions are unknown and no best heuristic solutions have been published, we give our suggested statistical bounds for them. In Table VII, we show two estimators and their corresponding intervals. We can see that the bounds for $\hat{\theta}_W$ are fairly tighter than that of $\hat{\theta}_{JK}^{(2)}$. By learning from the results of other three problems, the intervals of produced by $\hat{\theta}_{JK}^{(2)}$ are much reliable.

CONCLUDING REMARKS

In this paper, we studied two SOETs: truncation-point approach and extreme-value-theory when GRASP is applied on four combinatorial optimization problems. The optimal values are known for PMP, QAP and SCP. The point estimator and statistical bounds for them are constructed. We examined two estimators, $\hat{\theta}_{JK}^{(2)}$ and $\hat{\theta}_W$, for each problem instance. $\hat{\theta}_{JK}^{(2)}$ gives better estimation than $\hat{\theta}_W$ and the bounds of $\hat{\theta}_{JK}^{(2)}$ are reliable in terms of high coverage for PMP and SCP. However, less reliable bounds are found for QAP when GRASP heuristic is applied. For the 2-PNDP, we give our suggested interval estimation by using the SR criteria since the optimal value is unknown and studies on it are infrequent. Regarding the SR statistic that has been found important for judging the reliability of statistical bounds, we have had to content ourselves with SR in the range of 5 to 10 in this study of GRASP. This is a higher value possible indicating that the statistical bounds might be unreliable ($SR < 4$has been advocatedas a threshold for simulated annealing and vertex substitution for PMP [2], [8]). For problem instances with relatively low number of local searches, SR is difficult reduce to further without adding much more experiments. Hence, it remains an open question on how tight statistical bounds can be achieved by GRASP for the three combinatorial problems studied here with instances of known optimum.

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[28] The file is organized in the OR-Library and the test instances are available for downloading at http://people.brunel.ac.uk/~mastjjb/jeb/orlib/files/
[29] The data set is available by contacting the authors at mea@du.se.

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