DISPERSION OF AXISYMMETRIC LONGITUDINAL WAVES IN A SOLID CYLINDER MADE OF VISCOELASTIC MATERIAL

[1]SURKAY AKBAROV, [2]TARIK KOCAL, [3]TAMER KEPCELER

[1,2,3]Yildiz Technical University
E-mail: [1]akbarov@yildiz.edu.tr, [2]tkocal@yildiz.edu.tr, [3]kepceler@yildiz.edu.tr

Abstract—The paper studies the dispersion of axisymmetric longitudinal waves in the circular solid cylinder made of linear viscoelastic material. The investigations are carried out within the scope of the piecewise homogeneous body model by utilizing the exact equations of linear viscoelasto-dynamics. The material of the cylinder is described through fractional exponential operators and these relations are used in the numerical investigations. Dispersion curves are submitted for certain selected dispersive attenuation cases under various values of the problem parameters and the influence of the rheological parameters on these curves are discussed.

Keywords— Characteristic creep time, solid cylinder, Wave dispersion, Wave attenuation, Viscoelastic material.

I. INTRODUCTION

The study of time-harmonic wave dispersion and attenuation in viscoelastic material and in element of construction made from this material is required not only by theoretical, but also by application needs. The nondestructive inspection of tubes and pipes which are used in the infrastructure of many industries such as gas, oil, and water transport can be taken as an example of such applications. This is because, in many cases, these tubes are coated with viscoelastic polymer coatings for corrosion protection and therefore, under nondestructive testing of the tubes with guided waves, it is necessary to know the attenuation and dispersion rules of the waves propagating therein. Moreover, the study of the propagation of guided waves in viscoelastic materials and constructions made of viscoelastic elements, used in viscoelastic systems for attenuation of vibrations and waves caused by an earthquake or with various types of sound sources, is an another example of the application of this study.

Up to now, investigations related to wave propagation in structural elements made from viscoelastic materials have not been as numerous as studies which have been made for the same structural elements made from purely elastic materials. We consider a brief review of these investigations the first of which was made in the papers by Weiss [1] and Tamm and Weiss [2] which relate to Lamb wave propagation in an isotropic viscoelastic layer with stress-free surfaces. In these papers, non-dispersive attenuation is considered, i.e. it is assumed that the elastic constants are complex and independent of frequency.

Axisymmetric longitudinal guided wave dispersion and attenuation in a metal elastic hollow cylinder coated with a polymer viscoelastic layer was studied by Barshinger and Rose [3]. The viscoelasticity of the coated layer was taken into consideration through attenuation coefficients of the longitudinal and shear waves in the corresponding viscoelastic materials. These coefficients are determined experimentally for the frequencies in the order 1 - 5 MHz and are used for determination of the corresponding complex moduli. Consequently, using these complex moduli, the wave dispersion and attenuation dispersion in a metal elastic hollow cylinder coated with a polymer viscoelastic layer was investigated.

It follows from the works considered above, that the corresponding investigations on wave dispersion and attenuation were not connected with the more complicated and real models for viscoelastic materials and a few of the numerical results obtained in these works do not illustrate the character of the influence of the rheological parameters of the viscoelastic materials on this dispersion. In this sense, the first attempt was made in a paper by Akbarov and Kepceler [4] in which the torsional wave dispersion in the sandwich hollow cylinder made from linear viscoelastic materials was studied.

In the present paper, taking into consideration the foregoing discussions and significance of the related studies we continue the investigations begun by Akbarov and Kepceler [4] and attempt to investigate the axisymmetric longitudinal wave dispersion in a solid cylinder made of viscoelastic material, the rheological relations for which are given through the fractional exponential operators by Rabotnov [5]. The exact field equations and relations of the linear theory of viscoelasticity are used.

II. FORMULATION OF THE PROBLEM

We consider the solid circular cylinder shown in Fig. 1 and assume that the radius of the cylinder is R. We assume that the material of the cylinder is isotropic, homogeneous and hereditary-viscoelastic. We use the cylindrical system of coordinates $Or\theta z$ (Fig. 1) for determination of the position of the points of the system under consideration. Moreover, we assume that the cylinder has infinite length in the direction of

the Oz axis.

Thus, let us investigate the axisymmetric longitudinal wave propagation along the Oz axis in the considered solid cylinder with the use of the exactequations of motion of the linear theory for viscoelastic bodies.

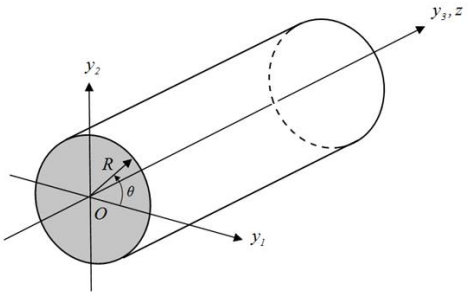


Fig. 1. Solid circular cylinder

We write the governing field equations and mechanical relations for the case under consideration.

Equations of motion:

$$\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{rz}}{\partial z}+\frac{1}{r}(T_{rr}-T_{\theta\theta})=\rho\ \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial T_{rz}}{\partial r}+\frac{\partial T_{zz}}{\partial z}+\frac{1}{r}T_{rz}=\rho\ \frac{\partial^2 u_z}{\partial t^2}. \qquad (1)$$

Constitutive relations:

$$T_{(ii)}=\lambda^{*}\theta\ +2\mu^{*}\varepsilon_{(ii)}\ ,\ (ii)=rr,zz,\theta\theta\ ,$$

$$T_{rz}=2\mu^{*}\varepsilon_{rz}\ ,\ \theta\ =\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz}\ , \qquad (2)$$

where λ^{*} and μ^{*} are the following operators:

$$\begin{Bmatrix}\lambda^{*}\\ \mu^{*}\end{Bmatrix}\varphi(t)=\begin{Bmatrix}\lambda_0\\ \mu_0\end{Bmatrix}\varphi(t)+\int_0^t\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(t-\tau)\varphi(\tau)d\tau \qquad (3)$$

In Eq. (3), λ_0 and μ_0 are the instantaneous values of Lame's constants as $t\to 0$, and $\lambda_1(t)$ and $\mu_1(t)$ are the corresponding kernel functions describing the hereditary property of the material.

Strain-displacement relations:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}\ ,\ \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),$$

$$\varepsilon_{\theta\theta}=\frac{u_r}{r}\ ,\ \varepsilon_{zz}=\frac{\partial u_z}{\partial z}. \qquad (4)$$

The relations (1) – (4) are the complete system of equations of the theory of linear viscoelasticity for isotropic bodies; conventional notation is used.

Consider also formulation of the boundary conditions. According to Fig. 1 we can write that

$$T_{rr}\Big|_{r=R}=0\ ,\ T_{rz}\Big|_{r=R}=0\ . \qquad (5)$$

This completes the formulation of the problem on the axisymmetric longitudinal wave dispersion in the solid cylinder made of viscoelastic material with arbitrary kernel functions $\lambda_1(t)$ and $\mu_1(t)$ which enter the constitutive relations (3).

III. METHOD OF SOLUTION

First, we represent the displacements and strains as:

$$u_r=v_r(r)e^{i(kz-\omega t)}\ ,\ u_z=v_z(r)e^{i(kz-\omega t)}\ ,$$

$$\theta\ =\upsilon\ (r)e^{i(kz-\omega t)}\ ,\varepsilon_{(ii)}=\gamma_{(ii)}(r)e^{i(kz-\omega t)},$$

$$(ii)=rr;\theta\theta;zz;rz \qquad (6)$$

where k is the wave number and ω is the circular frequency, and

$$\gamma_{rr}=\frac{dv_r(r)}{dr}\ ,\ \gamma_{\theta\theta}=\frac{v_r(r)}{r}\ ,\ \gamma_{zz}=\frac{dv_z(r)}{dz}\ ,$$

$$\gamma_{rz}=\frac{1}{2}\left(\frac{dv_r(r)}{dz}+\frac{dv_z(r)}{dr}\right), \qquad (7)$$

and we use the relation

$$\int_0^t f_1(t-\tau)f_2(\tau)d\tau\approx\int_{-\infty}^t f_1(t-\tau)f_2(\tau)d\tau\ , \qquad (8)$$

in the mechanical relations (2) and (3). Thus, taking the relations (6) – (8) into account in Eqs. (2) and (3), we can write the following relations:

Using the transformation $t-\tau=s$ we can make the following manipulations for the integrals which enter into Eq. (9):

$$\int_{-\infty}^t\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(t-\tau)e^{-i\omega\tau}d\tau=-\int_{\infty}^0\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(t-\tau)e^{-i\omega t}e^{i\omega s}ds=$$

$$e^{-i\omega t}\int_0^{\infty}\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(t-\tau)e^{i\omega s}ds=e^{-i\omega t}\left(\begin{Bmatrix}\lambda_{1c}\\ \mu_{1c}\end{Bmatrix}+i\begin{Bmatrix}\lambda_{1s}\\ \mu_{1s}\end{Bmatrix}\right) \qquad (10)$$

where

$$\begin{Bmatrix}\lambda_{1c}\\ \mu_{1c}\end{Bmatrix}=\int_0^{\infty}\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(s)\cos(\omega s)\,ds\ ,$$

$$\begin{Bmatrix}\lambda_{1s}\\ \mu_{1s}\end{Bmatrix}=\int_0^{\infty}\begin{Bmatrix}\lambda_1\\ \mu_1\end{Bmatrix}(s)\sin(\omega s)\,ds\ . \qquad (11)$$

Taking the relations (8) – (11) into account we can write the following expressions for the stresses:

$$T_{(ii)}=\Lambda\ \vartheta\ (r)e^{i(kz-\omega t)}+$$

$$2M\ \gamma_{(ii)}(r)e^{i(kz-\omega t)}=\sigma_{(ii)}(r)e^{i(kz-\omega t)},$$

$$T_{rz} = 2M\ \gamma_{rz}(r)e^{i(kz-\omega t)} = \sigma_{rz}(r)e^{i(kz-\omega t)} \quad , \quad (12)$$

where

$$\Lambda = \lambda_0 + \lambda_{1c} + i\lambda_{1s} \ , \ M = \mu_0 + \mu_{1c} + i\mu_{1s} \ . \quad (13)$$

Thus, we obtain the complex constants Λ and M (13), the real and imaginary parts of which are determined through the expressions (11) and (13). This means that the complete system of field Eqs. (1), (2), (4), (12) and (13) for the viscoelastic system, can also be obtained for the purely elastic system by replacing the elastic constants λ_0 and μ_0 with the complex constants Λ and M . In otherwords, the foregoing mathematical calculations confirm the dynamic correspondence principle [6] for the problem under consideration and the solution method used here coincides with this principle.

Thus, substituting the expression (12) into the equation of motion (1) and taking the relation (6) into consideration we obtain the following equations of motion in terms of the displacement amplitudes.

According to the foregoing transformations and expressions in (6) and (12), the boundary and contact conditions in (5) can be rewritten as follows:

$$\sigma_{rr}\Big|_{r=R} = 0 \ , \ \sigma_{rz}\Big|_{r=R} = 0 \, . \quad (16)$$

For the solution to the problem formulated through Eqs. (14), (6), (7) and (12), with the boundary and contact conditions in (16), according to Guz [7], we employ the following representation for the displacement amplitudes:

In (18), ζ_2 and ζ_3 are determined from the following equations:

where ω / k is the complex phase velocity of the wave propagation.

Thus, we determine the following expression for the function X from Eqs. (18) and (19).

$$X = A_1 J_0(\zeta_2 kr) + A_2 J_0(\zeta_3 kr) \quad (20)$$

where $J_0(x)$ is Bessel function of the first kind with zeroth order, respectively.

Using the expression (20) and Eqs. (17), (13), (6) and (7), we obtain the following dispersion equation from the conditions in (16):

$$\det \left\| \beta_{nm} \right\| = 0 \ , \ n; m = 1, 2 \quad (21)$$

where

In relation (22), $J_n(x)$ is Bessel function of the first kind, respectively. Moreover in (22) the following notation is used:

$$\chi = kR\zeta_2 \, ,$$

$$\chi_h = kR\zeta_3 \, , \beta_1 = \frac{(\Lambda(\omega) + 2M(\omega))}{(\Lambda(\omega) + M(\omega))},$$

$$\beta_2 = \frac{M(\omega)}{(\Lambda(\omega)+M(\omega))} - r\left(\frac{\omega}{k}\right)^2 (\Lambda(\omega)+M(\omega))^{-1}. \quad (23)$$

Thus, the dispersion equations obtained for the considered wave propagation problems have been derived in the form (21) – (23).

IV. NUMERICAL RESULTS AND DISCUSSIONS

According to the problem formulation, we must take the complex wave number k which can be presented as follows:

$$k = k_1 + ik_2 = k_1(1+i\beta),\ \beta = \frac{k_2}{k_1}, \quad (24)$$

where k_2 (or parameter β in (24)), i.e. the imaginary part of the wave number k, defines the attenuation of the wave amplitude under consideration and β is called the coefficient of the attenuation.

We determine the phase velocity of the studied waves through the expression

$$c = \frac{\omega}{k_1} \quad (25)$$

and introduce the notation $c_{20} = \sqrt{\mu_0/\rho}$.

We use below the arguments

$$\frac{c}{c_{20}},\ k_1R, \text{ and } \frac{h}{R}. \quad (26)$$

Thus, to solve the dispersion equation (21) it is necessary to give the values of λ_{1c}, λ_{1s}, μ_{1c} and μ_{1s} which are determined by the expressions in (11) through the kernel functions $\mu_1(t)$ and $\lambda_1(t)$ of the operators in (3). We recall that these operators are the viscoelastic property of the material of the cylinder. Consequently, for determination of the quantities λ_{1c}, λ_{1s}, μ_{1c} and μ_{1s}, it is necessary to give explicit expression for the functions $\mu_1(t)$ and $\lambda_1(t)$.

As in the papers by Akbarov [8] and Akbarov and Kepceler [4], here we also assume that the viscoelasticity of the material of the cylinder is described by Rabotnov's [5] fractional exponential operator, i.e. we assume that

$$\Pi_\alpha^*(x)\varphi(t) = \int_0^\infty \Pi_\alpha(x,t-\tau)\varphi(\tau)d\tau,$$

$$\Pi_\alpha(x,t) = t^{-\alpha}\sum_{p=0}^{\infty}\frac{(x)^p t^{p(1-\alpha)}}{\Gamma((1+n)(1-\alpha))}, 0 \le \alpha < 1. \quad (28)$$

In (28) $\Gamma(x)$ is the gamma function. Moreover, the constants α, β_0 and β_∞ in (27) and (28) are the rheological parameters of the solid cylinder. With these rheological parameters we can create dimensionless parameters in following.

Now we attempt to give mechanical sense to the parameter Q and d, although such meanings have been already discussed in the monograph by Akbarov [9] and in the papers by Akbarov [8] and Akbarov and Kepceler [4].

Thus, according to the foregoing discussions, we can conclude that the influence of the viscosity of the viscoelastic materials under consideration on the wave propagation velocity dispersion (i.e. on the dependence between c/c_{20} and k_1R) and on the wave attenuation dispersion (i.e. on the dependence between the attenuation coefficient β (24) and k_1R) can be characterized through the parameters Q and d. It must be taken into account that an increase in the values of the parameters Q and d will correspond to a decrease in the viscous part of the viscoelastic deformation of the constituent. Note that the influence of the other rheological parameter α on the viscous part of the viscoelastic deformation can be taken into account through the parameter Q (38).

This completes consideration of the selection of the dimensionless rheological parameters through which we will study the influence of the viscoelasticity properties of the cylinder material on the axisymmetric longitudinal wave dispersion.

Thus, according to Ewing et al. [10] and Kolsky [11], as in the paper by Akbarov and Kepceler [4], we assume that

$$\beta = \frac{1}{2}\frac{\mu_{1s}(\omega)}{\mu_0+\mu_{1c}(\omega)}, \quad (30)$$

First, we consider dispersion curves obtained for the homogeneous solid circular cylinder in the case where $\nu_0 = 0.3$ (instantaneous value of the Poisson coefficient) and $\alpha = 0.5$. These curves are constructed for the first lowest mode as a result of the solution to the corresponding dispersion equation which can be easily derived from the foregoing expressions and equations.

where

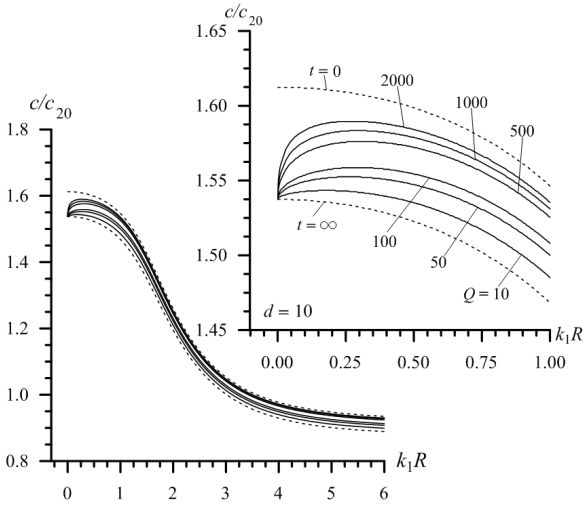


Fig. 2a. Dispersion curves constructed for the solid homogeneous cylinder under various values of the parameter Q under a fixed value of the parameter d (=10)

Corresponding results for the dispersive attenuation case where the coefficient of the attenuation β is determined through the expression (30) are given in Fig. 2.Note that the curves illustrated in this figure are constructed for various values of the parameter Q under a fixed value of the parameter d $(=10)$ (Fig. 2a) and for various values of the parameter d under a fixed value of the parameter $Q(=50)$ (Fig. 2b).

Corresponding dispersion curves for the coefficient of the attenuation β, i.e. the graphs of the dependents between β and the parameter Ω (38) are given in Fig. 3a for the case where $d=10$ and in Fig. 3b for the case where $Q=50$.

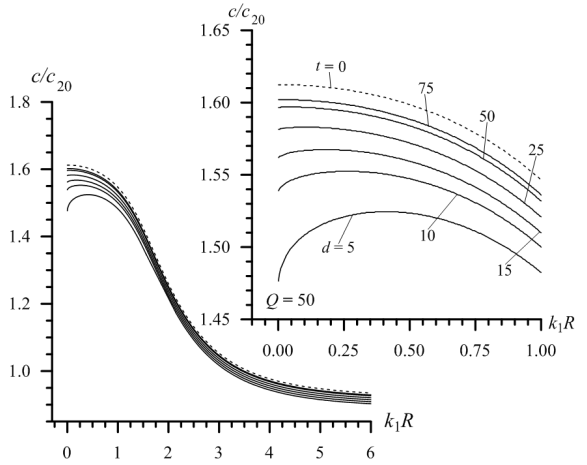


Fig. 2b. Dispersion curves constructed for the solid homogeneous cylinder under various values of the parameter d under a fixed value of the parameter Q (=50)

Thus, we turn to consideration of the numerical results given in Fig. 2, which show that they are limited with the velocities obtained for the instantaneous values (i.e. at $t=0$) of the elastic constants (upper limit) and with the velocities obtained for the long-term values (i.e. at $t=\infty$) of elastic constants (lower limit). It follows from Fig. 2 that the viscoelasticity of the cylinder material causes a decrease in the values of the wave propagation velocity.According to the foregoing discussions, it can be predicted that the low wavenumber limit of the wave propagation velocity depends only on the rheological parameter d.

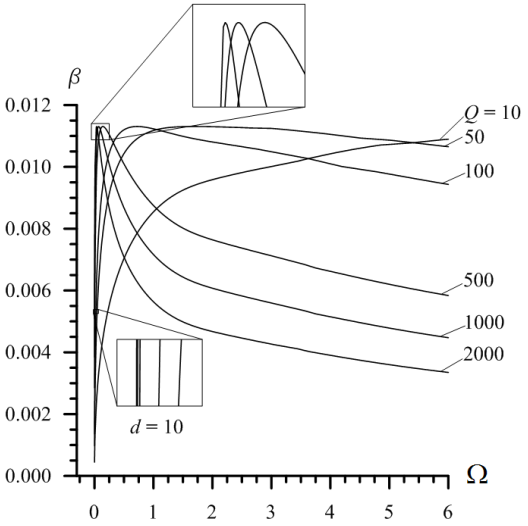


Fig. 3a. The graph of the dependence between the coefficient of the attenuation β and the dimensionless frequency Ω (38) constructed for various values of the parameter Q under a fixed value of the parameter d (=10)

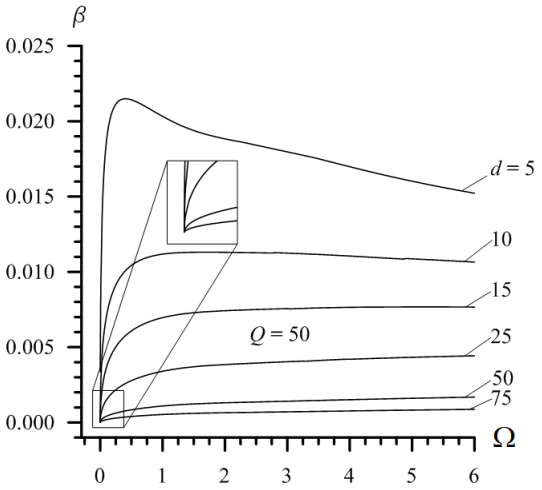


Fig. 3a. The graph of the dependence between the coefficient of the attenuation β and the dimensionless frequency Ω (38) constructed for various values of the parameter d under a fixed value of the parameter Q (=50)

Consequently, the low wavenumber limit values of the wave propagation velocity must be the same for various values of the rheological parameter Q under a fixed d and this limit is equal to the long-term bar velocity of the cylinder material for which the value E_∞ is determined through the expression $E_\infty=(1-1/(d+1))$ for the selected value of d .Note

that this prediction is proven by the results given in Fig. 2a. This prediction is also proven with the results illustrated in Fig. 2b, according to which, the low wavenumber limit values of the wave propagation velocity increase with the rheological parameter d.

Observation of the results allows us to conclude that, in the case under consideration the effect of the viscoelasticity of the cylinder material on the wave propagation velocity becomes significant under relatively small values of the dimensionless wavenumber $k_1 R$.

CONCLUSIONS

Thus, in the present paper, within the scope of the exact equations of motion of the theory of linear viscoelasticity, the axisymmetric longitudinal wave propagation in the solid cylinder made of viscoelastic material has been investigated. The corresponding dispersion equation is obtained for arbitrary hereditary type viscoelastic operators. For concrete numerical investigations, the viscoelasticity of the material is described by the fractional exponential operators by Rabotnov [5]. The dimensionless rheological parameters characterizing the characteristic creep time (denoted by Q) and the long-term values of the elastic constants (denoted by d) are introduced, and through these parameters the viscoelasticity of the constituent's material on the dispersion curves is studied. An increase in the values of these parameters means a decrease in the viscosity properties of the related material. The results related to the dispersive attenuation case are considered.

In the considered case, the viscoelasticity of the material of the cylinder causes the axisymmetric longitudinal wave propagation velocity to decrease. The magnitude of this decrease increases with a decrease in the aforementioned dimensionless rheological parameters.

The dispersion curves obtained for viscoelastic solid cylinder are limited by the dispersion curves obtained for the purely elastic case with instantaneous values of the elastic constants (upper limit) and by those obtained for the purely elastic case with long-term values of the elastic constants (lower limit);

A significant effect of the viscosity of the cylinder material on the wave propagation velocity appears in the case where $k_1 R \le 1.5$.

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