DEFINING EQUATIONS OF REDUCIBLE PLANE SEXTICS II

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Abstract - We showed the process to get the defining equations of a certain type of reducible plane sextics[3].For a homogeneous polynomial of degree six over the complex numbers, the vanishing set of the homogeneous polynomial in the complex projective plane P^2 is a curve that such curves depend on nineteen dimensional parameters. By using the Sylvester's elimination method, we show a certain type of the reducible plane sextics following the result that we performed earlier.

Keywords - Reducible plane sextics, Elimination method, Resultant.

I. INTRODUCTION

Let P^2 be a two-dimensional complex projective space with the coordinates [x, y, z], and let $f_6(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 6 in P^2. We consider the set $V_6 := \{(x, y, z) \mid f_6(x, y, z) = 0\}$. We call V_6 complex projective plane sextics (in short: plane sextics). The analytic set is a reducible plane sextic if it is factorized in C^3.
For a defining equation of an analytic set which has a singular point at the origin in C^3, there exist the following result[1].

Theorem 1
Let $f(z_0, z_1, z_2)$ be a polynomial in $C^3$and let V be an analytic set such that $V = \{(z_0, z_1, z_2) \mid f(z_0, z_1, z_2)= 0\}$ which has an isolated singular point at the origin in C^3. Then, for any i(i= 0, 1, 2), there exists an integer $a_i \geq 1$ and f has a term $z_i^{a}{}_i z_j$.

Using this theorem, we find the condition of the parameters of reducible plane sextics.

II. RESULTANT THEORY

Let $f_1, f_2, \ldots, f_n$ be elements of the polynomial ring $R=I[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n]$ in m+n variables over an integral domain I. If we wish to write a system of resultants explicitly, we can proceed as follows:

Regard the $f_i$as polynomials in $x_i$with coefficients in $I[x_2, \ldots, x_m, y_1, y_2, \ldots, y_n]$, and obtain resultants $R[f_i, f_j]$ by eliminating $x_1$from the pairs. Then eliminate $x_2$from these resultants, and so forth.
To obtain $R[f_i, f_j]$, we may use the following Sylvester's elimination method [2].

Theorem 2
Let f and g be polynomials in x with coefficients in I:
$f=a_0x^m+a_1x^{m-1}+\ldots+a_m$, $f=b_0x^n+b_1x^{n-1}+\ldots+b_n$
Let $R(f_i, f_j)$ be the determinant of matrix of degree m+n: [2]. Then $R(f_i, f_j)$ if and only if either f andghave a common root or $a_0 = b_0 =0$..

III. DEFINING EQUATION

In general, the defining equation of plane sextic with the coordinate [x, y, z] takes the following defining equation:

$$a_1x^6+(a_2y+a_3z)x^5+(a_4y^2+a_5yz+a_6z^2)x^4$$
$$+(a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3)x^3$$
$$+(a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+a_{15}z^4)x^2$$
$$+(a_{16}y^5+a_{17}y^4z+a_{18}y^3z^2+a_{19}y^2z^3+a_{20}yz^4+a_{21}z^5)x$$
$$+a_{22}y^6+a_{23}y^5z+a_{24}y^4z^2+a_{25}y^3z^3+a_{26}y^2z^4+a_{27}yz^5$$
$$+a_{28}z^6=0.$$

By suitable coordinate change, it takes the following defining equation:

$$x^5z+x^4yz+(y^3+b_1yz^2+b_2z^3)x^3$$
$$+(b_3y^3z+b_4y^2z^2+b_5yz^3+b_6z^4)x^2$$
$$+(b_7y^5+b_8y^4z+b_9y^3z^2+b_{10}y^2z^3+b_{11}yz^4+b_{12}z^5)x$$
$$+b_{13}y^6+b_{14}y^5z+b_{15}y^4z^2+b_{16}y^3z^3+b_{17}y^2z^4+b_{18}yz^5$$
$$+b_{19}z^6=0.$$

We show the process to get the defining equation of reducible plane sextics through the following steps.
Replacing x by $x' + \lambda z'$, y by $y' +\mu z'$, z by z', the following defining equation is as follows:

$$x^5z+x^4yz+c_1x^4z^2+(y^3+c_2y^2z+c_3yz^2+c_4z^3)x^3$$
$$+(c_5y^3z+c_6y^2z^2+c_7yz^3+c_8z^4)x^2$$
$$+(c_9y^5+c_{10}y^4z+c_{11}y^3z^2+c_{12}y^2z^3+c_{13}yz^4+c_{14}z^5)x$$
$$+c_{15}y^6+c_{16}y^5z+c_{17}y^4z^2+c_{18}y^3z^3+c_{19}y^2z^4+c_{20}yz^5$$
$$+c_{21}z^6=0.$$

Then

$$c_{14} = 5\lambda^4 + 4\lambda^3\mu + 3\lambda^3\mu^3 + 3\lambda^3\mu b_1 + 3\lambda^2 b_2 + 2\lambda\mu^3 b_3 + 2\lambda\mu^2 b_4 + 2\lambda\mu b_5 + 2\lambda b_6 + \mu^5 b_7 + \mu^4 b_8 + \mu^3 b_9 + \mu^2 b_{10} + \mu b_{11} + b_{12},$$

$$c_{20} = \lambda^4 + 3\lambda^3\mu^2 + \lambda^3 b_1 + 3\lambda^2\mu^2 b_3 + 2\lambda^2\mu b_4 + \lambda^2 b_5 + 5\lambda\mu^4 b_7 + 4\lambda\mu^3 b_8 + 3\lambda\mu^2 b_9 + 2\lambda\mu b_{10} + \lambda b_{11} + 6\mu^5 b_{13} + 5\mu^4 b_{14} + 4\mu^3 b_{15} + 3\mu^2 b_{16} + 2\mu b_{17} + b_{18},$$

$$c_{21} = \lambda^5 + \lambda^4\mu + \lambda^3\mu^3 + \lambda^3\mu b_1 + \lambda^3 b_2 + \lambda^2\mu^3 b_3 + \lambda^2\mu^2 b_4 + \lambda^2\mu b_5 + \lambda^2 b_6 + \lambda\mu^5 b_7 + \lambda\mu^4 b_8 + \lambda\mu^3 b_9 + \lambda\mu^2 b_{10} + \lambda\mu b_{11} + \lambda b_{12} + \mu^6 b_{13} + \mu^5 b_{14} + \mu^4 b_{15} + \mu^3 b_{16} + \mu^2 b_{17} + \mu b_{18} + b_{19}.$$

We rewrite the c_{14} and c_{21} by using c_{20}. And we take those c_{14}' and c_{21}'.

$$c_{14}' = \sum_{i=0}^{3} h_i \lambda^i, \quad c_{21}' = \sum_{i=0}^{3} g_i \lambda^i$$

whereh_i, g_i are polynomials of μ and $b_i (0 \le i \le 3)$.
By Sylvester's elimination method, we eliminate λ for c_{14}' and c_{21}'. Let $R(c_{14}', c_{21}')$ be the resultant of c_{14}' and c_{21}'. Then $R(c_{14}', c_{21}')$ is a polynomial of μ and $b_i (0 \le i \le 3)$.with degree 24.

$$R(c_{14}', c_{21}') = \sum_{i=0}^{24} k_i \lambda^i$$

wherek_i are polynomials of μ and $b_i (0 \le i \le 3)$.
Here,

$$k_{24} = -(1 + 25b_7 - 125b_{13})(4b_7^3 + 27b_{13}^3),$$

$$k_{23} = 5(4b_7^3 + 27b_{13}^3)(5b_3 - 4b_7 - 25b_8 + 20b_{13} + 125b_{14}).$$

Branch 1:

We put $b_7 := \frac{1}{25}(-1 + 125b_{13})$.

Then $k_{24} = 0$,

$$k_{23} = \frac{1}{390625}(2 + 125b_{13})^2(-1 + 500b_{13})(4 + 125b_3 - 625b_8 + 3125b_{14}).$$

Branch 2:

We put $b_{13} := -\frac{2}{125}$.

Then $k_{23} = 0$,

$$k_{22} = -\frac{27}{390625}(4 + 125b_3 - 625b_8 + 3125b_{14})^2.$$

We put $b_3 := \frac{1}{125}(-4 + 625b_8 - 3125b_{14})$.

Then $k_{22} = 0$ and $k_{21} = 0$,

$$k_{20} = -\frac{144}{9765625}(-49 + 7000b_1 - 37500b_4 - 8750b_8 - 390625b_8^2 + 187500b_9 + 87500b_{14} + 7812500b_8b_{14} - 39062500b_{14}^2 - 937500b_{15})(b_1 - 5b_4 + 25b_9 - 125b_{15}).$$

We put

$$b_1 := \frac{1}{7500}(49 + 37500b_4 + 8750b_8 + 390625b_8^2 - 187500b_9 - 87500b_{14} - 7812500b_8b_{14} + 39062500b_{14}^2 + 937500b_{15}).$$

Then $k_{20} = 0$,

$$k_{19} = -\frac{1}{152587890625}(7 + 625b_8 - 6250b_{14})^2(-637 + 1687500b_2 + 393750b_4 - 8437500b_5 - 52500b_8 + 35156250b_4b_8 + 5859375b_8^2 + 488281250b_8^3 - 3937500b_9 - 351562500b_8b_9 + 42187500b_{10} + 65625b_{14} - 351562500b_4b_{14} - 199218750b_8b_{14} - 18310546875b_8^2b_{14} + 3515625000b_9b_{14} + 1406250000b_{14}^2 + 219726562500b_8b_{14}^2 - 854492187500b_{14}^3 + 29531250b_{15} + 2636718750b_8b_{15} - 26367187500b_{14}b_{15} - 210937500b_{16}).$$

Branch 3:

We put $b_8 := \frac{1}{625}(-7 + 6250b_{14})$.

Then $k_{19} = 0$,

$$k_{18} = -\frac{3267}{15625}(b_2 - 5b_5 + 25b_{10} - 125b_{16})^2.$$

We put $b_2 := 5(b_5 - 5b_{10} + 25b_{16})$.

Then $k_{18} = 0$ and $k_{17} = 0$,

$$k_{16} = -\frac{441}{625}(b_6 - 5b_{11} + 25b_{17})(5b_4^2 + 12b_6 - 100b_4b_9 + 500b_9^2 - 60b_{11} + 750b_4b_{15} - 7500b_9 + 28125b_{15}^2 + 300b_{17}).$$

We put $b_6 := 5(b_{11} - 5b_{17})$.

Then $k_{16} = 0$,

$$k_{15} = \frac{693}{25}(b_4 - 10b_9 + 75b_{15})^2(b_{12} - 5b_{18}).$$

Branch 4:

We put $b_4 := 5(2b_9 - 15b_{15})$.

Then $k_{15} = 0$, $k_{14} = -\frac{7803}{25}(b_{12} - 5b_{18})^2$.

We put $b_{12} := 5b_{18}$.

Then $k_{14} = 0$ and $k_{13} = 0$,

$$k_{12} = -180b_{19}(b_5^2 - 20b_5b_{10} + 100b_{10}^2 + 150b_2b_{16} + 5625b_{16}^2 + 60b_{19}).$$

Branch 5:

We put $b_{19} := 0$.

Then $k_i = 0$ $(0 \le i \le 11)$.

Here,

$$b_7 = -\frac{3}{25}, \quad b_{13} = -\frac{2}{125}, \quad b_3 = -\frac{11}{125} + 25b_{14},$$

$$b_1 = 25(b_9 - 10b_{15}), \; b_8 = -\frac{7}{625} + 10b_{14},$$

$$b_2 = 5(b_5 - 5b_{10} + 25b_{16}), \quad b_1 = 5(b_{11} - 5b_{17}),$$

$$b_4 = 5(2b_9 - 15b_{15}), \quad b_{12} = 5b_{18}.$$

The plane sextic is reducible.

We back to Branch 5.
We put

$$b_{19} := \frac{1}{60}(-b_5^2 + 20b_5b_{10} - 100b_{10}^2 - 150b_5b_{14} + 1500b_{10}b_{14} - 5625b_{14}^2).$$

Then $k_{12}=0$,

$$k_{11} = \frac{9}{250}(b_5 - 10b_{10} + 75b_{16})^3(9b_5 - 90b_{10} - 750b_{11} + 3125b_5b_{14} - 31250b_{10}b_{14} + 675b_{16} + 234375b_{14}b_{16} + 7500b_{17}).$$

We put $b_5 := 5(2b_{10} - 15b_{16})$. Then $k_i = 0$ $(0 \le i \le 11)$.
Here,

$$b_7 = -\frac{3}{25}, \quad b_{13} = -\frac{2}{125}, \quad b_3 = -\frac{11}{125} + 25b_{14},$$
$$b_1 = 25(b_9 - 10b_{15}), b_8 = -\frac{7}{625} + 10b_{14},$$
$$b_2 = 25(b_{10} - 10b_{16}), \quad b_9 = 5(b_{11} - 5b_{17}),$$
$$b_4 = 5(2b_9 - 15b_{15}), \quad b_{12} = 5b_{18}.$$

The plane sextic is reducible.
As described above, we classify and confirm them.

III. RESULTS

We show some conditions of the parameters of the defining equation of reducible plane sextic. We showed the process to get the defining equations (Case1~8) of a certain type of reducible plane sextics [3]. We will show next cases.

$$b_7 = -\frac{3}{25}, b_{13} = -\frac{2}{125}, b_3 = -\frac{11}{125} + 25b_{14},$$
$$b_1 = \frac{279}{1250} + \frac{11}{2}b_{14} - \frac{3125}{4}b_{14}^2 - 25b_{15},$$
$$b_8 = -\frac{7}{625} + 10b_{14}, b_2 = -\frac{54}{78125} + \frac{500}{3}b_{14}^2 + \frac{390625}{9}b_{14}^3 + 9b_{15} + b_{14}(-\frac{3}{25} + 3125b_{15}) + 125b_{16},$$
$$b_6 = -\frac{1}{1318359375}(-54 + 9375b_{14} + 9765625b_{14}^2 + 703125b_{15})^2,$$
$$b_4 = \frac{279}{31250} + \frac{11}{5}b_{14} - \frac{625}{2}b_{14}^2 + 15b_{15}, b_{12} = 0, b_{19} = 0,$$
$$b_{11} = -\frac{2}{6591796875}(-54 + 9375b_{14} + 9765625b_{14}^2 + 703125b_{15})^2, b_{18} = 0,$$
$$b_5 = -\frac{108}{390625} + \frac{200}{3}b_{14}^2 + \frac{156250}{9}b_{14}^3 + \frac{18}{5}b_{15} + b_{14}(-\frac{6}{125} + 1250b_{15}) + 75b_{16},$$
$$b_{17} = -\frac{1}{32958984375}(-54 + 9375b_{14} + 9765625b_{14}^2 + 703125b_{15})^2,$$
$$b_9 = \frac{279}{312500} + \frac{11}{50}b_{14} - \frac{125}{4}b_{14}^2 + 9b_{15},$$
$$b_{10} = -\frac{54}{1953125} + \frac{20}{3}b_{14}^2 + \frac{15625}{9}b_{14}^3 + \frac{9}{25}b_{15} + b_{14}(-\frac{3}{625} + 125b_{15}) + 15b_{16}.$$

(Case 9)

$$b_7 = -\frac{3}{25}, b_{13} = -\frac{2}{125}, b_3 = -\frac{11}{125} + 25b_{14},$$
$$b_1 = 25(b_9 - 10b_{15}), b_8 = -\frac{7}{625} + 10b_{14},$$
$$b_2 = \frac{81}{125000} + \frac{1875}{8}b_{14}^2 + \frac{1953125}{72}b_{14}^3 - \frac{1}{6}b_9(9 + 3125b_{14}) + \frac{45}{2}b_{15} + \frac{1}{40}b_{14}(27 + 312500b_{15}) + 125b_{16},$$
$$b_6 = -\frac{1}{3375000000}(81 - 187500b_9 + 56250b_{14} + 9765625b_{14}^2 + 2812500b_{15})^2,$$
$$b_4 = 10b_9 - 75b_{15}, b_{12} = 0, b_{19} = 0,$$
$$b_{11} = -\frac{1}{8437500000}(81 - 187500b_9 + 56250b_{14} + 9765625b_{14}^2 + 2812500b_{15})^2, b_{18} = 0,$$
$$b_5 = \frac{81}{312500} + \frac{375}{4}b_{14}^2 + \frac{390625}{36}b_{14}^3 - \frac{1}{15}b_9(9 + 3125b_{14}) + 9b_{15} + b_{14}(\frac{27}{100} + 3125b_{15}) + 75b_{16},$$
$$b_{17} = -\frac{1}{84375000000}(81 - 187500b_9 + 56250b_{14} + 9765625b_{14}^2 + 2812500b_{15})^2,$$
$$b_{10} = \frac{81}{3125000} + \frac{75}{8}b_{14}^2 + \frac{78125}{72}b_{14}^3 - \frac{1}{150}b_9(9 + 3125b_{14}) + \frac{9}{10}b_{15} + b_{14}(\frac{27}{1000} + \frac{625}{2}b_{15}) + 15b_{16}.$$

(Case 10)

$$b_7 = -\frac{3}{25}, b_{13} = -\frac{2}{125}, b_3 = -\frac{11}{125} + 25b_{14}, \ b_1 = 25(b_9 - 10b_{15}),$$
$$b_8 = -\frac{7}{625} + 10b_{14}, b_2 = 5(b_5 - 5b_{10} + 25b_{16}), b_6 = 5(b_{11} - 5b_{17}),$$
$$b_4 = 10b_9 - 75b_{15}, b_{12} = 5b_{18}, b_{19} = 0.$$

(Case 11)

$$b_7 = -\frac{3}{25}, b_{13} = -\frac{2}{125}, b_3 = -\frac{11}{125} + 25b_{14},$$
$$b_1 = 25(b_9 - 10b_{15}), b_8 = -\frac{7}{625} + 10b_{14},$$
$$b_2 = 25(b_{10} - 10b_{16}), b_6 = 5(b_{11} - 5b_{17}),$$
$$b_{12} = 5b_{18}, b_4 = 10b_9 - 75b_{15}, b_{19} = 0, b_5 = 10b_{10} - 75b_{16}.$$

(Case12)

$$b_7 = -\frac{3}{25}, b_{13} = -\frac{2}{125}, b_3 = -\frac{11}{125} + 25b_{14},$$
$$b_1 = 25(b_9 - 10b_{15}), b_8 = -\frac{7}{625} + 10b_{14},$$
$$b_2 = 25(b_{10} - 10b_{16}), b_6 = \frac{5}{2}b_{11}, b_{12} = 5b_{18}, b_4 = 10b_9 - 75b_{15},$$
$$b_{19} = 0, b_{17} = \frac{1}{10}b_{11}, b_5 = 10b_{10} - 75b_{16}.$$

(Case 13)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=-\frac{11}{125}+25b_{14}$,

$b_1=25(b_9-10b_{15})$, $b_8=-\frac{7}{625}+10b_{14}$,

$b_2=25(b_{10}-10b_{16})$, $b_6=\frac{5}{2}b_{11}$, $b_{12}=0$, $b_4=10b_9-75b_{15}$,

$b_{19}=0$, $b_{17}=\frac{1}{10}b_{11}$, $b_{18}=0$, $b_5=10b_{10}-75b_{16}$.

(Case 14)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=-\frac{11}{125}+25b_{14}$,

$b_1=25(b_9-10b_{15})$, $b_8=-\frac{7}{625}+10b_{14}$,

$b_2=\frac{81}{125000}+\frac{1875}{8}b_{14}^2+\frac{1953125}{72}b_{14}^3-\frac{1}{6}b_9(9+3125b_{14})$
$+\frac{45}{2}b_{15}+\frac{1}{40}b_{14}(27+312500b_{15})+125b_{16}$,

$b_6=\frac{5}{2}b_{11}$, $b_{12}=0$, $b_4=10b_9-75b_{15}$, $b_{19}=0$, $b_{17}=\frac{1}{10}b_{11}$,

$b_{18}=0$, $b_5=\frac{81}{312500}+\frac{375}{4}b_{14}^2+\frac{390625}{36}b_{14}^3$
$-\frac{1}{15}b_9(9+3125b_{14})+9b_{15}+b_{14}(\frac{27}{100}+3125b_{15})+75b_{16}$,

$b_{10}=\frac{81}{312500}+\frac{75}{8}b_{14}^2+\frac{78125}{72}b_{14}^3-\frac{1}{150}b_9(9+3125b_{14})$
$+\frac{9}{10}b_{15}+b_{14}(\frac{27}{100}+625b_{15})+15b_{16}$.

(Case 15)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=\frac{4}{25}$, $b_1=\frac{4}{75}$, $b_8=\frac{11}{125}$, $b_2=-\frac{8}{3375}$,
$b_6=-\frac{16}{84375}$, $b_{12}=0$, $b_4=-\frac{4}{125}$, $b_{19}=0$, $b_{17}=-\frac{16}{2109375}$,
$b_{18}=0$, $b_5=\frac{8}{1875}$, $b_{11}=-\frac{32}{421875}$, $b_9=-\frac{12}{625}$, $b_{10}=\frac{56}{28125}$,
$b_{14}=\frac{31}{3125}$, $b_{15}=-\frac{4}{1875}$, $b_{16}=\frac{88}{421875}$.

(Case 16)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=\frac{4}{25}$, $b_1=\frac{8}{225}+\frac{1875}{22}b_{16}$,
$b_8=-\frac{125}{11}b_{16}$, $b_9=-\frac{2109375}{484}b_{16}^2$, $b_{12}=0$,
$b_4=-\frac{8}{375}-\frac{1125}{22}b_{16}$, $b_{19}=0$, $b_{17}=-\frac{84375}{484}b_{16}^2$,
$b_{18}=0$, $b_5=\frac{225}{11}b_{16}$, $b_{11}=-\frac{421875}{242}b_{16}^2$, $b_9=-\frac{8}{625}-\frac{725}{22}b_{16}$,
$b_{10}=\frac{105}{11}b_{16}$, $b_{14}=\frac{31}{3125}$, $b_{15}=-\frac{8}{5625}-\frac{725}{22}b_{16}$.

(Case 17)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=-\frac{11}{125}+25b_{14}$,

$b_1=\frac{279}{12500}+\frac{11}{2}b_{14}-\frac{3125}{4}b_{14}^2-25b_{15}$,

$b_8=-\frac{7}{625}+10b_{14}$, $b_2=-\frac{1}{4218750}(-51+3125b_{14})$
$(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})$,

$b_6=-\frac{1}{1318359375}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_{12}=0$, $b_4=\frac{279}{31250}+\frac{11}{5}b_{14}-\frac{625}{2}b_{14}^2+15b_{15}$, $b_{19}=0$,

$b_{17}=-\frac{1}{32958984375}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_{18}=0$, $b_5=-\frac{1}{2343750}(-11+3125b_{14})(-54+9375b_{14}$
$+9765625b_{14}^2+703125b_{15})$,

$b_{11}=-\frac{2}{6591796875}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_9=\frac{279}{312500}+\frac{11}{50}b_{14}-\frac{125}{4}b_{14}^2+9b_{15}$,

$b_{10}=-\frac{9}{390625}-\frac{78125}{18}b_{14}^3+b_{14}(\frac{7}{250}-\frac{625}{2}b_{15})+\frac{3}{10}b_{15}$,

$b_{16}=-\frac{1}{527343750}(3+21875b_{14})(-54+9375b_{14}+9765625b_{14}^2$
$+703125b_{15})$.

(Case 18)

$b_7=-\frac{3}{25}$, $b_{13}=-\frac{2}{125}$, $b_3=-\frac{11}{125}+25b_{14}$,

$b_1=\frac{279}{12500}+\frac{11}{2}b_{14}-\frac{3125}{4}b_{14}^2-25b_{15}$,

$b_8=-\frac{7}{625}+10b_{14}$, $b_2=-\frac{54}{4218750}+\frac{500}{3}b_{14}^2+\frac{390625}{9}b_{14}^3$
$+9b_{15}+b_{14}(-\frac{3}{25}+3125b_{15})+125b_{16}$,

$b_6=-\frac{1}{1318359375}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_{12}=0$, $b_4=\frac{279}{31250}+\frac{11}{5}b_{14}-\frac{625}{2}b_{14}^2+15b_{15}$, $b_{19}=0$,

$b_{17}=-\frac{1}{32958984375}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_{18}=0$, $b_5=-\frac{108}{390625}+\frac{200}{3}b_{14}^2+\frac{156250}{9}b_{14}^3+\frac{18}{5}b_{15}$
$+b_{14}(-\frac{6}{125}+1250b_{15})+75b_{16}$,

$b_{11}=-\frac{2}{6591796875}(-54+9375b_{14}+9765625b_{14}^2+703125b_{15})^2$,

$b_9=\frac{279}{312500}+\frac{11}{50}b_{14}-\frac{125}{4}b_{14}^2+9b_{15}$, $b_{10}=-\frac{54}{1953125}+\frac{20}{3}b_{14}^2$
$+\frac{15625}{9}b_{14}^3+\frac{9}{25}b_{15}+b_{14}(-\frac{3}{625}+125b_{15})+15b_{16}$.

(Case 19)

$$b_7=-\frac{3}{25},\ b_{13}=-\frac{2}{125},\ b_3=-\frac{11}{125}+25b_{14},\ b_1=25(b_9-10b_{15}),$$
$$b_8=-\frac{7}{625}+10b_{14},\ b_2=\frac{81}{125000}+\frac{1875}{8}b_{14}^2+\frac{1953125}{72}b_{14}^3$$
$$-\frac{1}{6}b_9(9+3125b_{14})+\frac{45}{2}b_{15}+\frac{1}{40}b_{14}(27+312500b_{15})+125b_{16},$$
$$b_6=-\frac{1}{3375000000}(81-187500b_9+56250b_{14}+9765625b_{14}^2$$
$$+2812500b_{15})^2,\ b_{12}=0,\ b_4=10b_9-75b_{15},\ b_{19}=0,$$
$$b_{17}=-\frac{1}{84375000000}(81-187500b_9+56250b_{14}+9765625b_{14}^2$$
$$+2812500b_{15})^2,\ b_{18}=0,\ b_5=\frac{81}{312500}+\frac{325}{4}b_{14}^2+\frac{390625}{36}b_{14}^3$$
$$-\frac{1}{15}b_9(9+3125b_{14})+b_{14}(\frac{27}{100}+3125b_{15})+75b_{16},$$
$$b_{11}=-\frac{1}{8437500000}(81-187500b_9+56250b_{14}+9765625b_{14}^2$$
$$+2812500b_{15})^2,\ b_{11}=\frac{81}{312500}+\frac{75}{8}b_{14}^2+\frac{78125}{72}b_{14}^3$$
$$-\frac{1}{150}b_9(9+3125b_{14})+\frac{9}{10}b_{15}+b_{14}(\frac{27}{1000}+\frac{625}{2}b_{15})+15b_{16}.$$

(Case 20)

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