(3, 2)-JECTION OPERATOR

NAVIN KUMAR SINGH

V.K.S., UNIVERSITY, Ara, Bihar (India)
E-mail: singhnavinkumar839@gmail.com

Abstract- This presentation focuses on a newly introduced linear operator called (3, 2)-jection operatorand attempts to throw some light on the relationship among projection,Trijection and(3,2)-jection operators. In this connection, we want to report a number of finding under certain conditions.

Index terms- Linear operator, Projection operator, Trijection operator, (3, 2)-jection operator.

I. INTRODUCTION

It is well known that projection plays a pivotal role in linear algebra and Functional analysis hereat we have taken care of the suitable generalization of this operator and introduced a new type of linear operator called a (3, 2)-jection operator.

II. IMPORTANT DEFINITIONS

(1) Linear operator: It is an operator E on a linear space L such that E(ax + by) = a E(x) + bE(y) $\forall\, x, y \in L$ and for scalars a and b.

(2) Projection operator : It is an operator E on some subspace M of a linear space L such that $E^2 = E$.

(3) Trijection operator : It is the linear operator E on a linear space L such that $E^3 = E$.

(4) (3, 2)-jection operator : It is the linear operator E on a linear space L such that $E^3 = E^2$.

III. MAIN RESULTS

Theorem 1 :
Let p and q be any two scalars such that p + q = 1 and E be a (3, 2)-jection then

$$(pE + qE^2)^n = E^2 \text{ for } 1 < n \in N$$

Proof : We denote the statement $(pE + qE^2)^n = E^2$ by P(n).

For n = 2, we have

$$(pE + qE^2)^2 = p^2E^2 + q^2E^4 + 2pqE^3$$
$$= p^2E^2 + q^2E^2 + 2pqE^2$$
$$\{\because E^3 = E^2 \,\&\, E^4 = E^2\}$$
$$= (p^2 + q^2 + 2pq)E^2$$
$$= (p+q)^2E^2$$
$$= E^2 \text{ (putting } p+q=1)$$

Hence P(2) is true

Now, we suppose that P(k) be true $\forall\ 2 \le k < N$

i.e. $(pE + qE^2)^k = E^2$ ….. (1.1)

For n = k + 1, we have

$$(pE + qE^2)^{k+1} = (pE + qE^2)^k (pE + qE^2)$$
$$= E^2(pE + qE^2) \quad \{\text{from (1.1)}\}$$
$$= pE^3 + qE^4$$
$$= pE^2 + qE^2$$
$$\{\text{since } E^3 = E^2, E^4 = E^2\}$$
$$= (p+q)E^2$$
$$= E^2 \text{ (putting } p+q=1)$$

Hence P(k) is true $\Rightarrow$ P(k + 1) is true. Thus by induction P(n) is true $\forall\ 2 \le n \in N$

Theorem 2:
If E be a trijection then

$$\left\{\frac{1}{2}(E + E^2)\right\}^n = \frac{1}{2}(E + E^2)\ \forall\ n \in N.$$

Proof : Since E is a trijection then from the definition, we have

$$E^3 = E \quad \text{….. (2.1)}$$

Now,

$$E^4 = E^3 \cdot E$$
$$= E \cdot E \quad \{\text{from (2.1)}\}$$
$$= E^2 \quad \text{….. (2.2)}$$

Here, we prove the theorem by the method of induction, so for convenience we denote the statement $"\left\{\frac{1}{2}(E + E^2)\right\}^n = \frac{1}{2}(E + E^2)"$ by P(n).

For n = 1, we have

$$\left\{\frac{1}{2}(E + E^2)\right\}^1 = \frac{1}{2}(E + E^2) \text{ which is clearly true.}$$

$\Rightarrow$ P(1) is true.

For n =2 , we have

$$\left\{\frac{1}{2}(E + E^2)\right\}^2 = \frac{1}{4}(E + E^2)^2$$
$$= \frac{1}{4}(E^2 + E^4 + 2E^3)$$
$$= \frac{1}{4}(E^2 + E^2 + 2E)$$
$$= \frac{1}{2}(E^2 + E) \quad \text{… (2.3)}$$

$\Rightarrow$ P(2) is true.

Now, we suppose P(k) be true $\forall\ 1 \le k < n \in N$ then

$$\left\{\frac{1}{2}(E+E^2)\right\}^k = \frac{1}{2}(E^2+E) \quad \dots (2.4)$$

For n = k + 1, we have

$$\left\{\frac{1}{2}(E+E^2)\right\}^{k+1} = \left\{\frac{1}{2}(E^2+E)\right\}^k \left\{\frac{1}{2}(E+E^2)\right\}$$

$$= \frac{1}{2}(E^2+E)\frac{1}{2}(E+E^2)$$

{from (2.4)}

$$= \left\{\frac{1}{2}(E+E^2)\right\}^2$$

$$= \frac{1}{2}(E+E^2) \qquad \{\text{from (2.3)}\}$$

Hence, P(k) is true $\Rightarrow$ P(k+1) is true.

So by the use of principal of induction P(n) is true $\forall$ n $\in$ N

Thus, $\left\{\frac{1}{2}(E+E^2)\right\}^n = \frac{1}{2}(E+E^2)$

In particular,

For n = 2, $\frac{1}{2}(E+E^2)$ is a projection

For n = 3, $\frac{1}{2}(E+E^2)$ is a trijection.

Also, $\left[\frac{1}{2}(E+E^2)^2\right]^3 = \frac{1}{2}(E+E^2)$ for n = 3

And $\left[\frac{1}{2}(E+E^2)^2\right]^2 = \frac{1}{2}(E+E^2)$ for n = 2

$\Rightarrow$ $\left[\frac{1}{2}(E+E^2)^2\right]^3 = \left[\frac{1}{2}(E+E^2)^2\right]^2$

$\Rightarrow$ $\frac{1}{2}(E+E^2)$ is a (3, 2) jection.

Theorem 3 :

If E be a trijection then $\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)$ $\forall$n $\in$ N.

Proof :

Since E is a trijection then from the definition, we have

$E^3 = E \quad \dots (3.1)$

and $E^4 = E^3.E$

$= EE \qquad \{\text{from (3.1)}\}$

$= E^2 \quad \dots (3.2)$

Now, we prove the theorem by the method of induction.

Here for convenience, we denote the statement "$\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)$ $\forall n \in N$, by P(n).

For n = 1, we have

$\frac{1}{2}(E^2-E) = \frac{1}{2}(E^2-E)$ which is trivial.

Hence P(1) is true.

For n = 2, we have

$$\left[\frac{1}{2}(E^2-E)\right]^2 = \frac{1}{4}(E^4+E^2-2E^3)$$

$$= \frac{1}{4}(E^2+E^2-2E)$$

{from (3.1) and (3.2)}

$$= \frac{1}{4}(2E^2-2E)$$

$$= \frac{1}{2}(E^2-E) \quad \dots (3.3)$$

$\Rightarrow$ P(2) is true.

Now, we suppose that P(k) be true $\forall 1 \le k < n \in N$ then

$$\left[\frac{1}{2}(E^2-E)\right]^k = \frac{1}{2}(E^2-E) \quad \dots (3.4)$$

For n = k + 1, we have

$$\left[\frac{1}{2}(E^2-E)\right]^{k+1} = \left[\frac{1}{2}(E^2-E)\right]^k \left[\frac{1}{2}(E^2-E)\right]$$

$$= \left[\frac{1}{2}(E^2-E)\right]\left[\frac{1}{2}(E^2-E)\right]$$

{from (3.4)}

$$= \frac{1}{2}(E^2-E) \qquad \{\text{from (3.3)}\}$$

So, P(k) is true $\Rightarrow$ P(k + 1) is true

Hence by the use of method of induction P(n) is true $\forall n \in N$.

i.e. $\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)$

In particular,

For n = 2, $\frac{1}{2}(E^2-E)$ is a projection

For n = 3, $\frac{1}{2}(E^2-E)$ is a trijection

Also, $\left[\frac{1}{2}(E^2-E)\right]^3 = \left[\frac{1}{2}(E^2-E)\right]^2$

So, $\frac{1}{2}(E^2-E)$ is a (3, 2)-jection.

Theorem 4. :

If E be a trijction then $(I-E)^n = I - E^2$ $\forall n \in N$.

Proof :

Since E is a trijection then from the definition, we have

$E^3 = E \quad \dots (4.1)$

and $E^4 = E^3.E$

$= EE \qquad \{\text{from (4.1)}\}$

$= E^2 \quad \dots (4.2)$

Now, we prove the theorem by the method of induction.

So, for convenience, we denote the statement
"$(I - E^2)^n = I - E^2 \ \forall n \in N$, by P(n).
For n = 1, we have
$I - E^2 = I - E^2$ which is trivial
$\Rightarrow$ P(1) is true
For n = 2, we have
$(I - E^2)^2 = I + E^4 - 2E^2$
$= I + E^2 - 2E^2$ {from (4.2)}
$= I - E^2$ … (4.3)
$\Rightarrow$ P(2) is true
Now, we suppose P(k) be true $1 \le k < n \in N$ then
$(I - E^2)^k = I - E^2$ … (4.4)
For n = k + 1, we have
$(I - E^2)^{k+1} = (I - E^2)^k (I - E^2)$
$= (I - E^2)(I - E^2)$ {from (4.4)}
$= (I - E^2)^2$
$= I - E^2$ {from (4.3)}
Thus, P(k) is true
$\Rightarrow$ P(k + 1) is true
So by applying the principle of induction
P(n) is true $\forall n \in N$
This means,
$(I - E^2)^n = I - E^2 \ \forall n \in N$
In particular,
For n = 2, $I - E^2$ is a projection
For n = 3, $I - E^2$ is a trijection
Also, $(I - E^2)^3 = (I - E^2)^2$
$\Rightarrow$ $I - E^2$ is a (3, 2)-jection
Theorem 5 :
If E be a trijection then

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I + \frac{1}{2}E - \frac{1}{2}E^2 \ \forall n \in N$$

Proof :
Since E is a trijection, then from the definition, we have
$E^3 = E$ … (5.1)
and $E^4 = E^3.E$
$= EE$ {from (5.1)}
$= E^2$ … (5.2)
Now, we prove the theorem by the method of induction.
So for convenience, we denote the statement

$$"\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I + \frac{1}{2}E - \frac{1}{2}E^2" \text{ by P(n).}$$

For n = 1, we have

$$I + \frac{1}{2}E - \frac{1}{2}E^2 = I + \frac{1}{2}E - \frac{1}{2}E^2 \text{ which is trivial.}$$

$\Rightarrow$ P(1) is true
For n = 2, we have

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^2 = I + \frac{1}{4}E^2 + \frac{1}{4}E^4 + E - \frac{1}{2}E^3 - E^2$$

$$= I + \frac{1}{4}E^2 + \frac{1}{4}E^2 + E - \frac{1}{2}E - E^2$$

{from (5.1) and (5.2)}

$$= I + \frac{1}{2}E - \frac{1}{2}E^2 \quad \ldots(5.3)$$

From (5.3), P(2) is true
Now, we suppose that P(k) be true for $k \in N$ such that $1 \le k \le n \in N$
then

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^k = I + \frac{1}{2}E - \frac{1}{2}E^2 \quad \ldots (5.4)$$

For n = k + 1,
we have

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^{k+1} = \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^k \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)$$

$$= \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)$$

{from (5.4)}

$$= \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^2$$

$$= I + \frac{1}{2}E - \frac{1}{2}E^2 \quad \{\text{from 5.3}\}$$

Thus, P(k) is true $\Rightarrow$ P(k + 1) is true
i.e. P(n) is true $\forall n \in N$ by the use of induction
Hence,

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^n = \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right) \ \forall n \in N$$

In particular,

For n = 2, $I + \frac{1}{2}E - \frac{1}{2}E^2$ is a projection

For n = 3, $I + \frac{1}{2}E - \frac{1}{2}E^2$ is trijection

Also,

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^3 = \left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^2$$

$\Rightarrow$ $I + \frac{1}{2}E - \frac{1}{2}E^2$ is a (3, 2)-jection
Theorem 6 :
If E be a trijection then

$$\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I - \frac{1}{2}E - \frac{1}{2}E^2 \ \forall n \in N$$

Proof : Since E is a trijection then
$E^3 = E$ … (6.1)
and $E^4 = E^3.E$
$= E \cdot E$ {from (6.1)}
$= E^2$ … (6.2)
Here, we prove the theorem by the use of method of induction.
Now for the convenience, we denote the statement

$$"\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I - \frac{1}{2}E - \frac{1}{2}E^2 \ \forall n \in N" \text{ by P(n)}$$

It is obvious that P(1) is true
For n = 2, we have

$$\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^2 = I + \frac{1}{4}E^2 + \frac{1}{4}E^4 - E + \frac{1}{2}E^3 - E^2$$

$$= I + \frac{1}{4}E^2 + \frac{1}{4}E^2 - E + \frac{1}{2}E - E^2$$

{from (6.1) & (6.2)}

$$= I - \frac{1}{2}E - \frac{1}{2}E^2 \qquad \ldots (6.3)$$

From (6.3), it is clear that P(2) is true

Now, we suppose that P(k) be true for $k \in N$ such that $1 \leq k < n \; \forall n \in N$

then

$$\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^k = I - \frac{1}{2}E - \frac{1}{2}E^2 \qquad \ldots (6.4)$$

For n = k + 1, we have

$$\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^{k+1} = \left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^k \left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)$$

$$= \left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)$$

{from (6.4)}

$$= \left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^2$$

$$= I - \frac{1}{2}E - \frac{1}{2}E^2 \quad \{\text{from (6.3)}$$

Hence, P(k) is true $\Rightarrow$ P(k + 1) is true

So, by the use of method of induction that P(n) is true $\forall n \in N$

Thus, $\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I - \frac{1}{2}E - \frac{1}{2}E^2$

In particular,

For n = 2, $I - \frac{1}{2}E - \frac{1}{2}E^2$ is a projection

For n = 3, $I - \frac{1}{2}E - \frac{1}{2}E^2$ is a trijcetion

Also, $\left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^3 = \left(I - \frac{1}{2}E - \frac{1}{2}E^2\right)^2$

$\Rightarrow \; I - \frac{1}{2}E - \frac{1}{2}E^2$ is a (3,2)-jection.

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