THERMO-MAGNETIC PROPERTIES OF THE SYNTHETICALLY CHARGED INTERACTING NEUTRAL BOSON ATOMS IN 2D DEEP OPTICAL LATTICE

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Abstract— In this research, the thermo-magnetic properties of the synthetically charged interacting neutral boson atoms in 2D deep optical lattice are studied. The temperature dependence of the magnetization and magnetic susceptibility are investigated. The analysis relied on the Hartree-Fock approximation where an integral over the phase space is performed to calculate the accurate grand canonical thermo-magnetic potential. Subsequently, the grand potential is used to calculate the above mentioned parameters. Thereby, the effect of interaction- interaction and optical potential depth is considered. The obtained results provide important magnetic properties**.**

Index Terms—Dynamic or Topological Ultracold Gases, Trapped Gases Diamagnetism, Paramagnetism, And Super-Paramagnetism.

INTRODUCTION

Studying the temperature dependence of the magnetic properties, including magnetization and magnetic susceptibility, of magnetic condensed matter systems is hindered by the complex structure and interactions present in such systems. The difficulty of controllable varying system parameters make it difficult to capture the physics of such systems with simple models. This limitation can be circumvented, physicist have devised a new way to simulate these system [1], using ultra-cold boson atoms in a state of Bose-Einstein condensation (BEC) and degenerate. Fermi gases. These ultra cold systems provides controllable model that can be simulate the behavior of many quantum systems [2]. In this sprite, quantum magnetism constitutes an important aspect of the general quantum simulation program based on rotating BEC in optical lattice [3], such as ferro and anti-ferromagnetism [4], Bose glass [5], and spinor system in optical lattice [6], etc. So, it is important to calculate the relevant thermo-magnetic parameters for this system under realistic experimental condition, in order to be able to analyze existing experimental data and interpret these results from a theoretical viewpoint.

Rotating BEC in 2D optical lattice is an important system for studying several fundamental problems in condensed matter physics [7]. A unique property of optical lattices, as realized in experiments, is that the periodic lattice is accompanied by a rotating harmonic confining potential, arising from an external magnetic trap or in applying Raman lasers to the gas to realize a Berrys phase for a moving particle. So, this system can demonstrate various effects caused by the magnetic field such as quantum Hall effect [8] and the superfluid to Mott insulator phase transition [9]. For deep optical lattices, the system can be mapped onto the model of Josephson junctions arrays in presence of an external magnetic field [10].

Experimentally, Bloch and his coworker have demonstrated a new type of optical lattice that realizes strong effective magnetic fields for rotating condensate boson atoms [3,11] inspired by the proposal of Jaksch and Zoller [7] and subsequent work of Gerbier [12]. In the present work, we consider this system theoretically as in ordinary quantum statistical mechanics. We directly calculate the magnetization and magnetic susceptibility of an interacting synthetically charged neutral boson atoms. The above mentioned two parameters are calculated from the relevant grand canonical potential (q-potential) for this system. The accurate q-potential is calculated using the Hartree-Fock approximation [13–16]. Our obtained results showed that, the thermo-magnetic properties depend strongly on the interatomic interaction, as well as optical potential depth.

THEORETICAL MODEL

In experiments with rotating atomic gases in optical lattice [10,17,18], particles are usually trapped by a harmonic potential, $V_{trap}(r_\perp, z) = \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)$, with $r_\perp^2 = (x^2 + y^2)$ and stirred by time-varying magnetic field or auxiliary laser beams. In the frame rotating at angular speed $\Omega = \Omega\hat{z}$, the Hamiltonian of a single particle can be written as:

$$H = \frac{|P_\perp - m\Omega \times r_\perp|^2}{2m} + \frac{P_z^2}{2m} + V_{ext}(r_\perp, z) \qquad (1)$$

Where

$$V_{ext}(r_\perp, z) = V_{rot}(r_\perp, z) + V_{opt}(r_\perp) \qquad (2)$$

$$V_{rot}(r_\perp, z) = \frac{1}{2}m[(\omega_\perp^2 - \Omega^2)r_\perp^2 + \omega_z^2 z^2]$$

$$V_{opt}(r_\perp) = V_0[sin^2(kx) + sin^2(ky)]$$

With $\{\omega_\perp, \omega_z\}$ are the frequencies of the harmonic potential along the coordinate directions. V_0 is the lattice potential depth and k = 2π/λ′ is the wave vector

of the laser light, λ′ is the laser wavelength. In terms of k, one has to introduce the recoil energy $E_R = (\hbar^2 k^2/2m)\{\equiv \hbar\omega_R\}$ as an energy scale for specifying the lattice depth. It's defined as the recoil energy that one atom requires when it absorbs one lattice photon.
It is interesting to see how one can implemented exactly the magnetic Hamiltonian, $H_{mag} = \frac{1}{2m}[p_\perp - qA(r)]^2$, where $A(r)$ is the vector potential associated to the field B and $p_\perp$ the momentum operator of the particle, with this rotation method. If we suppose that all relevant energies are much smaller than $\hbar\omega_z$, the motion along the z direction is frozen and the problem is effectively two dimensional in agreement with the Hamiltonian H_{mag} Hmag. In the limit of centrifugal deconfinement $\boldsymbol{\Omega} \to \omega_\perp$, the system is formally equivalent to bosons of charge q^*=1 in uniform magnetic field B^* , provided that we choose the symmetric gauge $A = (B \times r)/2$. Thus the effective vector potential and the synthetic magnetic field is defined in such that $q^*A(r) = m\Omega \times r \perp$, justifying the analogy between the Coriolis and the Lorentz forces. These consideration enabled us to rewrite $V_{rot}(r_\perp, z)$ in Eq.(3) in the form

$$V_{rot}(r_\perp, z) \equiv V_{syn}(r_\perp, z) = \frac{1}{2}m[\omega_\perp^2(1 - B^2)r_\perp^2 + z^2] \quad (3)$$

Note that, the axis and speed of rotation represents the direction and strength of the synthetic magnetic field.
Limitations in the experimental techniques, due to the difficulty of centrifugal deconfinement, have so far prevented researchers from reaching higher artificial magnetic fields. However, in the presence of the optical lattice centrifugal suppression) is extinguished. For shallow optical lattice, the effect of optical potential is to act as a stirring mechanism . As well as for the deeper lattice depth, atoms may be only localized in lattice sites. Consequently, the effect of centrifugal force is controlled in both cases. In a deep optical lattice where the boson atoms are localized at the optical potential minima and hopping between different lattice sites is negligible. An approximated expression for the optical potential can be readily obtained by extending the standard harmonic approximation for the optical lattice potential V_{lat}[7]. Under this approximation we have to write

$$V_{opt}(r_\perp) \equiv V_{lat}(r_\perp) = \frac{m}{2}\omega_{lat}^2(x^2 + y^2) = \frac{m}{2}\omega_{lat}^2 r_\perp^2 \quad (4)$$

with on-site trapping frequency,

$$\omega_{lat}^2 = \frac{4sE_R^2}{\hbar^2}, s = \frac{V_0}{E_R} \text{ s}$$

is a dimensionless parameter accounted for the optical potential depth in terms of the recoil energy. Blakie et. al.[19] pointed that the optical potential have a unique property, as realized in experiments, is that the periodic lattice is accompanied by a harmonic confining potential. Thus the external confinement potential becomes

$$V_{ext}^*(r_\perp, z) = V_{syn}(r_\perp, z) + V_{lat}(r_\perp, z) = \frac{1}{2}m[\omega_\perp^2(1 - B^2 + ss_\perp)r_\perp^2 + \omega_z^2 z^2] \quad (5)$$

is the confinement potential for synthetically charged atoms in deep optical lattice, where $s_\perp = \omega_R/\omega_\perp$ is the ratio between recoil frequency of the optical potential and the radial frequency of the harmonic potential (we shall refer to this ratio as the reduced recoil frequency).

Finally, the effective Hamiltonian of interacting neutral boson atoms synthetically charged using the method
of rotating the cloud in optical lattice is given by

$$H = \frac{\left|P_\perp - m\omega_\perp B \times r\perp\right|^2}{2m} + \frac{P_z^2}{2m} + V_{ext}^*(r_\perp, z) \quad (6)$$

In order to embody the interatomic interaction effect in our approach, we follow the Hadzibabic [20,21] and Dalfovo [22] approach and consider the mean field approximation. The simplest scheme in this approximation is the Hartree-Fock theory.

The relevant thermo-magnetic parameters, include the magnetization, the magnetic susceptibility and heat capacity at constant magnetic field can be calculated using the logarithm of the grand canonical partition function for the system described by the Hamiltonian (6), we will refer to it by the thermo-magnetic potential [23,24].

A. Accurate thermo-magnetic potential for synthetically charged condensate boson atoms

Usually, BEC is described within the grand canonical ensemble, the thermo-magnetic potential q [25,26] is given
By :

$$q(B,T) = -\sum_{n=0}^{\infty} \ln\left(1 - e^{-\beta(E_n - \mu(B))}\right) \quad (7)$$

where $\beta = 1/(k_B T)$, E_n is the eigenvalues for the potential (11) and μ(B) is the chemical potential for the neutral boson atoms put in synthetic magnetic field. For later convenient we have to separate out the ground state contribution and expand the logarithm, $\ln(1 - y) = -\sum_{j=1}^{\infty} \frac{y^j}{j}$, in this case Eq.(10) can be rewritten as:

$$q(B,s,T) = q_0(B,T) + \sum_j \frac{Z^j}{j} \sum_{n=1}^{\infty} e^{-j\beta E_n} \equiv q_0(B,T) + q_{th}(B,T) \quad (8)$$

with $q_B = -\ln(1 - z)$ is the grand potential for the synthetically charged atoms, q_{th} is the grand potential for thermal atoms and $z = e^{\beta(\mu(B) - E_0)}$ is the effective fugacity.

The sum in Eq.(8) cannot be evaluated analytically in a closed form. Another possible way to do this analysis, is to approximate the sum by integral (semi classical approximation) over the phase space per unit volume of the phase space, $(2\pi\hbar)^3$. In order to do this, we replace the discrete eigenvalue E_n with a continuous variable $\epsilon(r,p)$ depends on the position r and the momentum p,

$$\epsilon(r_\perp,p_\perp)=\frac{|P_\perp-m\omega_\perp B\times r_\perp|^2}{2m}+\frac{p_z^2}{2m}+V_{ext}^*(r_\perp,z)$$

which corresponds to the classical energy associated with the single-particle Hamiltonian for the system, finally, $q_{th}(B,T)$ in Eq.(8) can be written as :

$$q_{th}(B,s,T)=-\frac{1}{(2\pi\hbar)^3}\sum_{j=1}^{\infty}\frac{Z^j}{j}\int d^2p_\perp dp_z\, d^2r_\perp dz \times e^{-j\beta[\frac{|p_\perp-m\omega_\perp B\times r_\perp|^2}{2m}+\frac{p_z^2}{2m}+V_{ext}^*(r_\perp,z)]} \quad (9)$$

After doing the p integration by making the change of variables $p_\perp\rightarrow p_\perp-m\omega_\perp B\times r_\perp$, the integral in Eq.(12) takes the same form as in the absence of synthetic magnetic field with an effective frequencies $\sqrt{\omega_\perp^2(1-B^2)}$and ω_z, respectively. In this case the local thermo-magnetic grand potential becomes:

$$q_{th}(B,T)=\frac{1}{\lambda_{th}^3}\sum_{j=1}^{\infty}\frac{Z^j}{j^{5/2}}\int e^{-j\beta V_{eff}(B,T)}d^2\,r_\perp dz \quad (10)$$

where $\lambda_{th}=\sqrt{\frac{2\pi\hbar^2}{mk_BT}}$ is the thermal de-Broglie wavelength.

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B. Hartree-Fock approximation for deep optical potential depth

In the self-consistent Hartree-Fock approach, we consider the synthetically charged and non-synthetically charged atoms as a multiphase fluid model (better known as the two fluid model designating two phases or components). The two fluid model treats the general case of modelling each phase or component as a separate fluid with its own set of governing balance equations. For the system under consideration in this work the non- synthetically charged atoms are treated as a non-interacting gas with density $n_{th}(r_\perp,z)$ confined by the effective potential $V_{eff}(r_\perp,z)$ given by the combination of the trapping potential $V_{ext}^*(r_\perp,z)$and the self-consistent interaction potential $2g\big(n_T(r_\perp)+n_0(r_\perp)\big)$, i.e. Under this approximation the effective trapping potential is given by :

$$V_{eff}(r_\perp,z)=V_{ext}^*(r_\perp,z)+2g[n_T(r_\perp,z)+n_0(r_\perp,z)\big) \quad (11)$$

The densities of the non-charged and the synthetically charged component are given as a solution of the two coupled equations: the non-charged atoms (thermal) satisfy Schrödinger's equation:

$$[\frac{|P_\perp-m\omega_\perp B\times r_\perp|^2}{2m}+\frac{p_z^2}{2m}+V_{eff}(r_\perp,z)]\psi_i(r_\perp,z)=\varepsilon_i\psi_i(r_\perp,z) \quad (12)$$

and the synthetically charged (condensate) part satisfies the time independent Gross-Pitaevskii equation,

$$\Big[\frac{|P_\perp-m\omega_\perp B\times r_\perp|^2}{2m}+\frac{p_z^2}{2m}+V_{eff}(r_\perp,z)-gn_0(r_\perp,z)\Big]\emptyset(r_\perp,z)=\mu(B,s)\emptyset(r_\perp,z) \quad (13)$$

Thomas-Fermi solution for Eq.(13) provides the density of the synthetically charged atoms as,

$$n_0(r_\perp,z)=\frac{\mu(B,s)-V_{ext}^*(r_\perp,z))}{g}\theta(\mu(B,s)-V_{ext}^*(r_\perp,z)) \quad (14)$$

where $\theta(x))$ is the unit positive step function. Within the same approximation, if the effect of non-charged atoms on the synthetically charged atoms is neglected, Eq.(11) becomes :

$$V_{eff}(r_\perp,z)=V_{ext}^*(r_\perp,z)+2gn_0(r_\perp,z),\\ =|V_{ext}^*(r_\perp,z)-\mu(B,s)|+\mu(B,s) \quad (15)$$

where Eq.(14) is used here. Since in our approximation both the effect of synthetically magnetic field and the optical potential lead to a shift for the harmonic oscillator confinement, then the shape of the confining potential has a three-dimensional Mexican-hat shape [27,28]. Moreover, $\mu(B,s)$ is the relevant energy scale parametrization the effects of interactions [29–32], up to the point in the trap where $(B,s)=V_{syn}^*(B,s)$. Furthermore, the normalization condition $\int n_0((r_\perp,z))dr_\perp dz$ yields the chemical potential,

$$\mu(B,s)=\mu(0)(1-B^2sS_\perp)^{\frac{2}{5}} \quad (16)$$

where $\mu(0)=\frac{1}{2}\hbar\omega_g(\frac{15N_0a}{a_{har}})^{\frac{2}{5}}$ is the chemical potential for the harmonically trapped atoms, a is the s-wave scattering length, $a_{har}=\sqrt{\hbar/m\omega_g}$ and $\omega_g=(\omega_\perp^2)^{\frac{1}{3}}$. The parametrized $\mu(B,V_0)$ in Eq.(16) look like a generalization for the well-knownThomas-Fermi approximation of harmonically trapped boson [22]. Moreover can be set

in terms of Dalfovo's interaction scaling parameter $\eta \equiv \frac{\mu_0}{k_B T_B}$ [22] which fixed the chemical potential of the harmonically trapped gas in units of the transition temperature for the non-interacting gas in the same trap. In terms of η Eq.(16) becomes :

$$\mu(B, V_0) = \eta k_B T_0 (1 - B^2 + 4ss_\perp)^{2/5} \qquad (17)$$

with,

$$T_0 = \frac{\hbar\omega_g}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}(1 - B^2 + 4ss_\perp)^{1/3} \qquad (18)$$

is the BEC transition temperature for the synthetically charged ideal boson atoms.

When the temperature of the system is relatively high (compared with μ(B,s) kBT) the majority of non-charged atoms (thermal atoms) lie outside the synthetically charged atoms (condensate atoms) in the region where

$$V^*_{eff}(r_\perp, z) - \mu(B, s) > 0$$

and

$$V^*_{eff}(r_\perp, z) = V^*_{ext}(r_\perp, z)$$

these two conditions are seen clearly from Eq.(15). Following the same procedure outlined in Ref.([16]) we have:

$$q(B, s, T) = q_0 + \frac{1}{(1-B^2+4ss_\perp)}\{\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_4(z) + \frac{\mu((B,s))}{k_B T}\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_3(z)\} \qquad (19)$$

with $g_k(z) = \sum_{j=1}^{\infty}\left(\frac{z^j}{j^k}\right)$ is the usual Bose function. Using the grand potential q, the total number of synthetically charged particles is given by

$$N = N_0 + \frac{1}{(1-B^2+4ss_\perp)}\{\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_3(z) + \frac{\mu((B,s))}{k_B T}\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_2(z)\} \qquad (20)$$

II. MAGNETIZATION AND MAGNETIC SUSCEPTIBILITY

The synthetic magnetization can be evaluated using the relation $M = -\frac{1}{\beta}\left(\frac{\partial q(B,s,T)}{\partial B}\right)_{\beta,\mu}$, thus:

$$M = -\frac{1}{\beta}\left(\frac{\partial q_0}{\partial B}\right)_{\beta,\mu} - \left(\frac{\partial}{\partial B}\right)_{\beta,\mu}\frac{1}{\beta}\frac{1}{(1-B^2+4ss_\perp)}\{\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_4(z) + \frac{\mu((B,s))}{k_B T}\left(\frac{k_B T}{\hbar\omega_g}\right)^3 g_3(z)\} \qquad (21)$$

However magnetization have two different phases as a function of temperature. It is well-known that, for a given number of atoms, magnetization exhibits positive value above and below BEC transition temperature. Therefore, for $T < T_0$ from equations (21), one has

$$\frac{\bar{M}_{\tau<1}}{Nk_B T_0} = \frac{2\bar{B}}{(1-B^2+4ss_\perp)}[\mathcal{T}^4\frac{\zeta(4)}{\zeta(3)} + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}\mathcal{T}^3] \qquad (22)$$

where $\tau = \frac{T}{T_0}$ is the normalized temperature. On the other hand, for τ > 1, we have the total magnetization per particle as:

$$\frac{\bar{M}_{\tau>1}}{Nk_B T_0} = \frac{2\bar{B}}{(1-B^2+4ss_\perp)}[\mathcal{T}^4\frac{g_4(z)}{\zeta(3)} + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}\frac{g_3(z)}{\zeta(3)}\mathcal{T}^3 \qquad (23)$$

Equations (22) and (23) reveal the characteristic behavior of the synthetic magnetization as a function of B and T. It's clear that the magnetization vanishes in the absence of the synthetic magnetic field, as well as at $T = 0$ the total magnetization is zero. For non-zero synthetic magnetic field the net magnetization $M \neq 0$ for any temperature greater than zero.

Finally, to justify the magnetic state of the system we have to calculate the magnetic susceptibility at constant synthetic magnetic field. An expression for the magnetic susceptibility can be derived by inserting Eq.(21) into $\chi = \left(\frac{\partial M}{\partial \bar{B}}\right)_T$. Below the transition temperature, the magnetic susceptibility of the system reads:

$$\frac{\chi_{\tau<1}}{Nk_B T} = \frac{2(1+3\bar{B}^2+4ss_\perp)}{(1+\bar{B}^2+4ss_\perp)^3}[\mathcal{T}^4\frac{\zeta(4)}{\zeta(3)} + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}\mathcal{T}^3] \qquad (24)$$

while for τ > 1 the magnetic susceptibility given by:

$$\frac{\chi_{\tau>1}}{Nk_B T} = \frac{2(1+3\bar{B}^2+4ss_\perp)}{(1+\bar{B}^2+4ss_\perp)^3}[\mathcal{T}^4\frac{g_4(z)}{\zeta(3)} + \eta\left(1 - B^2 + 4ss_\perp\right)^{\frac{2}{5}}\frac{g_3(z)}{\zeta(3)}\mathcal{T}^3\Big] + \frac{2\bar{B}}{(1-B^2+4ss_\perp)}[\mathcal{T}^4\frac{g_3(z)}{\zeta(3)} + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}\frac{g_2(z)}{\zeta(3)}\mathcal{T}^3] \times \frac{1}{z}\frac{dz}{dB} \qquad (25)$$

With

$$\frac{1}{z}\frac{\partial z}{\partial \bar{B}} = -\frac{[g_3(z) + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}g_2(z)\mathcal{T}^{-1}]}{[g_2(z) + \eta(1 - B^2 + 4ss_\perp)^{\frac{2}{5}}g_1(z)\mathcal{T}^{-1}]}$$

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DISCUSSION AND CONCLUSION

Fig. 1 illustrate the relation between the synthetic magnetization versus the reduced temperature for different values of the synthetic magnetic field strength. Above the critical temperature, τ > 1 the

system possess magnetization irrelevant to temperature but as temperature lowered below T0, the magnetization decreases monotonically to a plateau at very low temperature. Finally reaches to zero as τ decreases to zero.

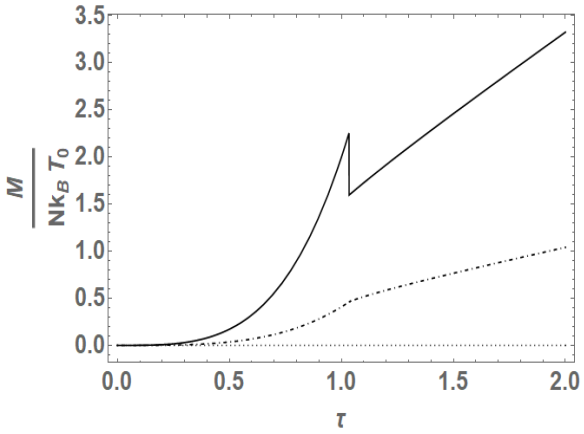


Fig. 1. The synthetic magnetization versus reduced temperature for different values of the synthetic field strength, $\bar{B}$ = 0.0 dot-line, $\bar{B}$ = 0.5 dotdashed-line, and $\bar{B}$ = 1.0 solid-line

In Fig. 2 the magnetic susceptibility versus reduced temperature for different magnetic field strength is given. This figure shows that the magnetic susceptibility has a negative value approximately constant at high temperature and strong synthetic magnetic field, i.e, the system is in diamagnetic phase. At moderate synthetic magnetic field field, the temperature dependence of the synthetic susceptibility displays paramagnetic behavior at temperature greater than BEC transition temperature T_0 and ferromagnetic phase at temperature less than T_0. At τ = 1, the synthetic susceptibility has a jump. The behavior of the susceptibility as B is increased is of interest. Increasing the value of B tends to clear up the susceptibility maximum and to broaden the curves.

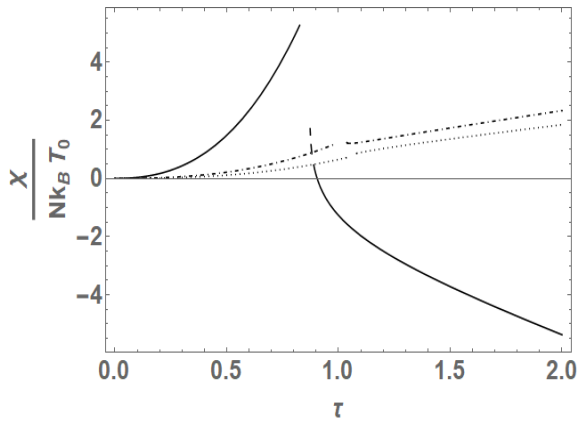


Fig. 2. The synthetic magnetic susceptibility versus reduced temperature for $\bar{B}$ = 0.0 dot-line, $\bar{B}$ = 0.5 dotdashed-line, and$\bar{B}$ = 1.0 solid-line.

In conclusion, we have presented the results of a study of the synthetically charged Bose gas in a 2D optical lattice. For certain values of the temperature and synthetic magnetic field we have obtained analytical results for the magnetization and susceptibility. We emphasize that the present analysis shows that the use of the appropriate simulator is a useful and complementary tool for exploring and learning about the physics of magnetic quantum system.

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