SOMOS'S DEDEKIND N-FUNCTION IDENTITIES OF LEVEL 6 AND ITS APPLICATIONS TO COLORED PARTITIONS

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Abstract- M. Somos discovered many Dedekind n-function identities of different levels using computer and o_ered no proof for them. The purpose of this paper is to prove these Dedekind n-function identities of level 6 and to establish certain colored partitions.

Keywords- Theta functions, Modular equations, Dedekind eta-function, colored partitions. 2010 Mathematics Subject Classi_cation: 11F20, 33C05, 11P83.

I. INTRODUCTION

The dedekind n-function identity is defined as

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = e^{\pi i\tau/12}(1-e^{2n\pi i\tau}), \qquad \mathrm{Im}(\tau) > 0.$$

Throughout this paper, we assume $|q| < 1$ and use the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

By Jacobi's triple product identity [3, p. 35], we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

S. Ramanujan defines the following special cases of $f(a,b)$ [3, p. 36]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty =: \theta_3(0,2z),$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

Also after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty.$$

Note that, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for $\mathrm{Im}(\tau) > 0$. The theta-function identity, which relates $f(-q)$ to $f(-q^n)$ is called theta function identity of level n. Ramanujan recorded several identities which involves $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ in his second notebook [6] and 'Lost' notebook [7]. For example [4, p. 204, Entry 51],

$$f_1^8f_2^8f_3^4f_6^4 + 9q^3f_1^4f_2^4f_3^8f_6^8 = q^3f_1^{12}f_6^{12} + q^2f_2^{12}f_3^{12}. \tag{1.1}$$

where $f_n = f(-q^n)$. Note that these modular equations are also called Schläfli-type. Since the publication of [3], several authors including N. D. Baruah [1, 2], M. S. M. Naika [5] and K. R. Vasuki [10] have found additional modular equations of the type (1.1). Recently M. Somos [8] discovered around 6200 theta-function identities of different levels and these identities highly resembles Ramanujan's recordings. He runs PARI/GP scripts to check these identities and offers no proof for them. Furthermore, B. Yuttanan [12] has proved certain Somos's theta-function identities of different levels, by using Ramanujan's modular equations and deduced certain interesting partition identities for them. K. R. Vasuki and R. G. Veeresha [11] have proved Somos's theta-function identities of level fourteen and B. R. Srivatsa Kumar and R. G. Veeresha [9] have obtained partition identities for these theta-function identities. Somos [8] conjectured around 100 new elegant η-function identities of level six by using PARI/GP scripts without offering proof. These identities have the arguments in $f(-q), f(-q^2), f(-q^3)$ and $f(-q^6)$, namely $-q, -q^2, -q^3$ and $-q^6$ all have exponents dividing six, which is thus equal to the 'level' of the identity six. The purpose of this paper is to prove new η-function identities of level six conjectured by Somos and to establish partition-theoretic interpretation. Before proceeding to state and prove Somos's identities, we first recall certain modular equations and theta-function identities which we need in sequel.

We now define a modular equation as defined by Ramanujan. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \qquad |z|<1,$$

denotes an ordinary hyper-geometric function with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Then, we say that β is of degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

On page 230 of his second notebook [6] and [3, pp. 230−238, Entry 5], Ramanujan recorded the following modular equation of degree 3. If β has degree three over α and m is the multiplier of degree three, then

$$m = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1 - 2(\alpha\beta)^{1/4}}, \tag{1.2}$$

$$\frac{3}{m} = \frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1}{1 - 2(\alpha\beta)^{1/4}} \tag{1.3}$$

and if $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$, then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0. \tag{1.4}$$

From (1.2) and (1.3), we have

$$\frac{m^2}{3} = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1} \tag{1.5}$$

Transcribing (1.4) and (1.5) into theta functions by employing [3, pp. 122−124, Entry 10(i) and Entry 12(v)], we deduce

$$\frac{q\chi^6(q)}{\chi^6(q^3)} + \frac{\chi^6(q^3)}{\chi^6(q)} = \chi^3(q)\chi^3(q^3) - \frac{8q}{\chi^3(q)\chi^3(q^3)} \tag{1.6}$$

and

$$\frac{\varphi^4(q)}{3\varphi^4(q^3)} = \frac{1 - 4q\frac{\chi^3(q)}{\chi^9(q^3)}}{\frac{4\chi^3(q^3)}{\chi^9(q)} - 1}. \tag{1.7}$$

Also one can easily see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{1.8}$$

From (1.8), we observe that

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{\chi^2(q)}{\chi^2(q^3)}\frac{f_2}{f_6}. \tag{1.9}$$

Before concluding this section, for convenience we set

$$a := a(q) = q^{-1/24}\chi(q) \qquad \text{and} \qquad b := b(q) = q^{-1/8}\chi(q^3).$$

2 Dedekind η-function identities of level six

Theorem 2.1. *We have*

$$f_1^{12} f_2^5 f_3 f_6 + 72q^2 f_1^3 f_2^2 f_3^4 f_6^{10} + 8q f_1^7 f_2^6 f_6^6 - f_1^{13} f_3^6 - 9q f_2^5 f_3^{13} f_6 = 0.$$

Proof. Multiplying (1.6) throughout by $(ab)^{-24}(8a^{18}b^6 - a^{15}b^{15} - 64a^{12} - 16a^9b^9 + 64b^{12})$, we obtain

$$\frac{b^3}{a^9} - \frac{16}{a^6b^6} - \frac{112}{b^{15}a^3} + \frac{64}{b^{24}} - \frac{512}{a^{21}b^9} + \frac{128}{a^{12}b^{12}} - 8\frac{a^6}{b^{18}} - \frac{64}{a^{24}} + \frac{80}{a^{15}b^3} + 9\frac{a^3}{b^9} - 1 + \frac{512}{b^{21}a^9} = 0,$$

which is equivalent to

$$\frac{3b^3}{a^9}\left(1 - \frac{4a^3}{b^9}\right)^3 + \left(\frac{8a^6}{b^{18}} + \frac{a^3}{b^9}\right)\left(\frac{4b^3}{a^9} - 1\right)^3 - \left(1 + \frac{8}{a^6b^6}\right)\left(1 - \frac{4a^3}{b^9}\right)^2\left(\frac{4b^3}{a^9} - 1\right) = 0.$$

Employing (1.7) in the above, we see that

$$\frac{b^3}{a^9}\frac{\varphi^4(q)}{\varphi^4(q^3)}+9\left\{\frac{8a^6}{b^{18}}+\frac{a^3}{b^9}\right\}\frac{\varphi^8(q^3)}{\varphi^8(q)}-\frac{8}{a^6b^6}-1=0.$$

Using (1.9) in the above, we obtain

$$\frac{1}{ab^5}\left(\frac{f_2}{f_6}\right)^4+9\left\{\frac{8}{a^{10}b^2}+\frac{b^7}{a^{13}}\right\}\left(\frac{f_8}{f_2}\right)^8-\frac{8}{a^6b^6}-1=0.$$

On replacing q to $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.8) and then multiplying throughout by $f_1^{13}f_3^6$, we deduce the result. □

Theorem 2.2. *We have*

$$f_1^5f_2^{12}f_3f_6+8qf_2^{13}f_6^6-f_1^6f_2^7f_3^6-9qf_1^2f_2^3f_3^{10}f_6^4-9q^2f_1^5f_3f_6^{13}=0.$$

Proof. Multiplying (1.6) throughout by $a^{-15}b^{-24}(a^{18}b^3-4a^9b^6-a^6b^{15}-64a^3+32b^9)$, we obtain

$$\frac{4a^3}{b^9}-\frac{b^3}{a^9}-\frac{44}{a^6b^6}+\frac{64}{b^{15}a^3}-\frac{64}{b^{24}}+1+\frac{4a^6}{b^{18}}+\frac{32}{a^{15}b^3}+\frac{192}{a^{12}b^{12}}-\frac{512}{b^{21}a^9}+\frac{a^{15}}{b^{21}}-\frac{a^{12}}{b^{12}}=0,$$

which is equivalent to

$$\frac{3b^3}{a^9}\left(1-\frac{4a^3}{b^9}\right)^3-\left(1+\frac{8}{a^6b^6}\right)\left(1-\frac{4a^3}{b^9}\right)^2\left(\frac{4b^3}{a^9}-1\right)-\left(\frac{a^{15}}{b^{21}}-\frac{a^{12}}{b^{12}}\right)\left(\frac{4b^3}{a^9}-1\right)^3=0.$$

Employing (1.7) in the above, we see that

$$\frac{b^3}{a^9}\frac{\varphi^4(q)}{\varphi^4(q^3)}-\frac{8}{a^6b^6}-1-9\left(\frac{a^{15}}{b^{21}}-\frac{a^{12}}{b^{12}}\right)\frac{\varphi^8(q^3)}{\varphi^8(q)}=0.$$

Using (1.9) above, we obtain

$$\frac{1}{ab^5}\left(\frac{f_2}{f_6}\right)^4-\frac{8}{a^6b^6}-1-9\left(\frac{1}{ab^5}-\frac{b^4}{a^4}\right)\left(\frac{f_6}{f_2}\right)^8=0.$$

On replacing q to $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.8) and then multiplying throughout by $f_1^6f_2^7f_3^6$, we deduce the result. □

Theorem 2.3. *We have*

$$f_1f_2^{13}f_3^5+72q^2f_2^6f_6^{13}-f_1^{10}f_2^4f_3^2f_6^2-81q^2f_1f_2f_3^5f_6^{12}-9qf_1^6f_3^6f_6^7=0.$$

Proof. Multiplying (1.6) throughout by $a^{-12}b^{-27}(a^{15}b^6-32a^9+4a^6b^9-a^3b^{18}+64b^3)$, we obtain

$$\frac{b^3}{a^9}+\frac{4}{a^6b^6}+\frac{64}{b^{15}a^3}+\frac{192}{b^{24}}-1+\frac{4a^3}{b^9}-\frac{44a^6}{b^{18}}+\frac{32a^9}{b^{27}}-\frac{64}{a^{12}b^{12}}-\frac{512}{b^{21}a^9}+\frac{a^{12}}{b^{12}}-\frac{a^{15}}{b^{21}}=0,$$

which is equivalent to

$$\left(\frac{b^3}{a^9}-1\right)\left(1-\frac{4a^3}{b^9}\right)^3+\left(\frac{8a^6}{b^{18}}+\frac{a^{12}}{b^{12}}\right)\left(\frac{4b^3}{a^9}-1\right)^2\left(1-\frac{4a^3}{b^9}\right)-3\frac{a^{15}}{b^{21}}\left(\frac{4b^3}{a^9}-1\right)^3=0.$$

Employing (1.7) in the above, we see that

$$\left(\frac{b^3}{a^9}-1\right)+9\left(\frac{8a^6}{b^{18}}+\frac{a^{12}}{b^{12}}\right)\frac{\varphi^8(q^3)}{\varphi^8(q)}-1-81\frac{a^{15}}{b^{21}}\frac{\varphi^{12}(q^3)}{\varphi^{12}(q)}=0.$$

Using (1.9) in the above, we obtain

$$\frac{b^3}{a^9}-1+9\left(\frac{8}{a^{10}b^2}+\frac{b^4}{a^4}\right)\left(\frac{f_6}{f_2}\right)^8-81\frac{b^3}{a^9}\left(\frac{f_6}{f_2}\right)^{12}=0.$$

On replacing q to $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.8) and then multiplying throughout by $f_1^{10}f_2^4f_3^2f_6^3$, we deduce the result. □

Theorem 2.4. *We have*

$$f_1^{13}f_2f_6^5+72qf_2^6f_3^7f_6^6+8f_1^4f_2^{10}f_3^3f_6^2-81qf_1f_2f_3^{12}f_6^5-9f_1^6f_3^{13}=0.$$

Proof. Multiplying (1.6) throughout by $(ab)^{-24}(a^{15}b^{15}-64a^{12}+16a^9b^9-8a^6b^{18}+64b^{12})$, we obtain

$$\frac{81b^3}{a^9}-\frac{144}{a^6b^6}+\frac{720}{b^{15}a^3}-\frac{576}{b^{24}}-\frac{72b^6}{a^{18}}-\frac{1008}{a^{15}}b^3+\frac{1152}{a^{12}b^{12}}-\frac{4608}{b^{21}a^9}+\frac{576}{a^{24}}+\frac{4608}{a^{21}b^9}-9+\frac{9a^3}{b^9}=0,$$

which is equivalent to

$$9\left(\frac{b^3}{a^9}+\frac{8b^6}{a^{18}}\right)\left(1-\frac{4a^3}{b^9}\right)^3-9\left(\frac{8}{a^6b^6}+1\right)\left(\frac{4b^3}{a^9}-1\right)^2\left(1-\frac{4a^3}{b^9}\right)+27\frac{a^3}{b^9}\left(\frac{4b^3}{a^9}-1\right)^3=0.$$

Employing (1.7) in the above, we see that

$$\left(\frac{b^3}{a^9}+\frac{8b^6}{a^{18}}\right)\frac{\varphi^8(q)}{\varphi^8(q^3)}-9\left(\frac{8}{a^6b^6}+1\right)+81\frac{a^3}{b^9}\frac{\varphi^4(q^3)}{\varphi^4(q}=0.$$

Using (1.9) in the above, we obtain

$$\left(\frac{a^7}{b^{13}}+\frac{8}{a^2b^{10}}\right)\left(\frac{f_2}{f_6}\right)^8-\frac{72}{a^6b^6}-9+\frac{81}{a^5b}\left(\frac{f_6}{f_2}\right)^4=0.$$

On replacing q to $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.8) and then multiplying throughout by $f_1^6f_3^{13}$, we deduce the result. □

III. APPLICATIONS TO COLORED PARTITIONS

The identities proved in Section 2 have applications to the theory of partitions. In this section, we present partition-theoretic interpretations of some of the theta- function identities proved in the previous section. For simplicity, we adopt the standard notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty,$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \qquad (r < s); r, s \in \mathbb{N}.$$

For example, $(q^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty$ which is $(q^2; q^8)_\infty\ (q^6; q^8)_\infty$.

Definition 3.1. A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions.
For example, if 1 is allowed to have 2 colors, then all the colored partitions of 2 are $2,\ 1_r + 1_r,\ 1_g + 1_g$ and $1_r + 1_g$, where we use the indices r (red) and g (green) to distinguish the two colors of 1. Also

$$\frac{1}{(q^a; q^b)_\infty^k},$$

is the generating function for the number of partitions of n, where all the parts are congruent to $a \pmod b$ and have k colors.

Theorem 3.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with one and two colors respectively and $+3 \pmod 6$ with thirteen colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with ten and fourteen colors respectively and $+3 \pmod 6$ with nineteen colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with six colors each and $+3 \pmod 6$ with nineteen colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 6$ with six colors and $+3 \pmod 6$ with seven colors. Let $p_5(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with thirteen and fourteen colors respectively and $+3 \pmod 6$ with thirteen colors. Then for any positive integer $n \geq 2$ the following identity holds:*

$$p_1(n) + 72p_2(n-2) + 8p_3(n-1) - p_4(n) - 9p_5(n-1) = 0.$$

Proof. Theorem 2.1 is equivalent to

$$\frac{1}{(q_1^{1\pm}, q_2^{2\pm}, q_{13}^{3+}; q^6)_\infty} + \frac{72q^2}{(q_{10}^{1\pm}, q_{14}^{2\pm}, q_{19}^{3+}; q^6)_\infty} + \frac{8q}{(q_6^{1\pm}, q_6^{2\pm}, q_{19}^{3+}; q^6)_\infty} - \frac{1}{(q_6^{2\pm}, q_7^{3+}; q^6)_\infty} - \frac{9q}{(q_{13}^{1\pm}, q_{14}^{2\pm}, q_{13}^{3+}; q^6)_\infty} = 0.$$

The five quotients of the above identity represent the generating functions $p_1(n), p_2(n), p_3(n), p_4(n)$ and $p_5(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + 72q^2 \sum_{n=0}^{\infty} p_2(n)q^n + 8q \sum_{n=0}^{\infty} p_3(n)q^n - \sum_{n=0}^{\infty} p_4(n)q^n - 9q \sum_{n=0}^{\infty} p_5(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = p_5(0) = 1$. Now on equating the coefficients of of q^n in the above, we are lead to the desired result.

Example: The following table verifies the case for $n = 2$ in the above theorem.

$p_1(2) = 3$:	$1_r + 1_r, 2_g, 2_y.$
$p_2(0) = 1$:	
$p_3(1) = 6$:	$1_r, 1_g, 1_y, 1_o, 1_b, 1_p.$
$p_4(1) = 6$:	$2_r, 2_g, 2_y, 2_o, 2_b, 2_p.$
$p_5(1) = 13$:	$1_r, 1_g, 1_y, 1_o, 1_b, 1_p, 1_i, 1_{gr}, 1_{bl}, 1_m, 1_v, 1_{pu}, 1_c$

□

Theorem 3.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm2 \pmod 6$ with one and two colors respectively and $+3 \pmod 6$ with ten colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm, \pm2 \pmod 6$ with six colors each and $+3 \pmod 6$ with sixteen colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm2 \pmod 6$ with six colors and $+3 \pmod 6$ with four colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm2 \pmod 6$ with four and fourteen colors respectively and $+3 \pmod 6$ with four colors. Let $p_5(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm2 \pmod 6$ with one and fourteen colors and $+3 \pmod 6$ with ten colors. Then for any positive integer $n \geq 2$ the following identity holds:*

$$p_1(n) + 8p_2(n-1) - p_3(n) - 9p_4(n-1) - 9p_5(n-2) = 0.$$

Proof. Theorem 2.2 is equivalent to

$$\frac{1}{(q_1^{1\pm}, q_2^{2\pm}, q_{10}^{3+}; q^6)_\infty} + \frac{8q}{(q_6^{1\pm}, q_6^{2\pm}, q_{16}^{3+}; q^6)_\infty} - \frac{1}{(q_6^{2\pm}, q_4^{3+}; q^6)_\infty} - \frac{9q}{(q_4^{1\pm}, q_{14}^{2\pm}, q_4^{3+}; q^6)_\infty} - \frac{9q^2}{(q_1^{1\pm}, q_{14}^{2\pm}, q_{10}^{3+}; q^6)_\infty} = 0.$$

The five quotients of the above identity represent the generating functions for $p_1(n), p_2(n), p_3(n), p_4(n)$ and $p_5(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + 8q\sum_{n=0}^{\infty} p_2(n)q^n - \sum_{n=0}^{\infty} p_3(n)q^n - 9q\sum_{n=0}^{\infty} p_4(n)q^n - 9q^2\sum_{n=0}^{\infty} p_5(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = p_5(0) = 1$. Now on equating the coefficients of of q^n in the above, we are lead to the desired result.

Example: The following table verifies the case for $n = 3$ in the above theorem.

$p_1(3) = 13:$	$1_r + 1_r + 1_r, 1_r + 2_g, 1_r + 2_g, 3_r, 3_g, 3_y, 3_o, 3_p, 3_b, 3_{bl}, 3_c, 3_{gr}, 3_i.$
$p_2(2) = 27:$	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_p + 1_p, 1_b + 1_b, 1_r + 1_g, 1_r + 1_y$ $1_r + 1_o, 1_r + 1_p, 1_r + 1_b, 1_g + 1_y, 1_g + 1_o, 1_g + 1_p, 1_g + 1_b, 1_y + 1_o,$ $1_y + 1_p, 1_y + 1_b, 1_o + 1_p, 1_o + 1_b, 1_p + 1_b, 2r, 2_g, 2_o, 2_p, 2_b, 2_y.$
$p_3(3) = 4:$	$3_r, 3_g, 3_o, 3_b.$
$p_4(2) = 24:$	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_r + 1_g, 1_r + 1_y, 1_r + 1_o, 1_g + 1_y$ $1_g + 1_o, 1_y + 1_o, 2_r, 2_g$ and 12 others of the same type.
$p_5(1) = 1:$	1_r

□

As the pattern of proof of next two theorems are identical with our proof of Theorem 3.1 and 3.2, we just give the references of the required identities.

Theorem 3.3. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with nine colors each and $+3 \pmod 6$ with ten colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with ten and seventeen colors respectively and $+3 \pmod 6$ with sixteen colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 6$ with nine colors and $+3 \pmod 6$ with four colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with nine and twenty one colors respectively. Let $p_5(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm 2 \pmod 6$ with four and seventeen colors respectively and $+3 \pmod 6$ with four colors. then for any positive integer $n \geq 2$ the following identity holds:*

$$p_1(n) + 72p_2(n-2) - p_3(n) - 81p_4(n-2) - 9p_5(n-1) = 0.$$

Proof. To prove Theorem 3.3, we employ Theorem 2.3. □

Theorem 3.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 6$ with nine colors and $+3 \pmod 6$ with thirteen colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pm 2 \pmod 6$ with thirteen and seventeen colors respectively and $+3 \pmod 6$ with thirteen colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pm 2 \pmod 6$ with nine colors each and $+3 \pmod 6$ with nineteen colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pm 2 \pmod 6$ with twelve and twenty one colors respectively and $+3 \pmod 6$ with thirteen color. Then for any positive integer $n \geq 1$ the following identity holds:*

$$p_1(n) + 72p_2(n-1) + 8p_3(n) - 81p_4(n-1) - 9p_5(n) = 0.$$

Proof. To prove Theorem 3.4, we employ Theorem 2.4. □

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