APPROXIMATION OF RADIAL BASIS FUNCTION WITH LINEAR POLYNOMIAL: COMPARISON AND APPLICATION

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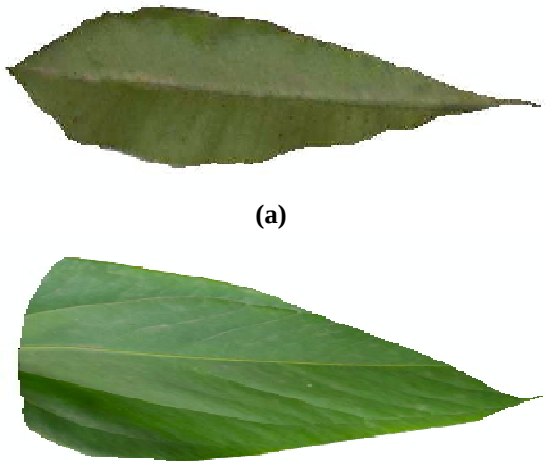
Abstract - In this research paper we proposed an interpolation approach to model a continuous differntiable surfaces of a real leaf from 3D scanned data points. As such model is essential for modellig the plant architecture, fluid movement and biological structure such as photosynthesis. The interpolation approach is based on adding a linearpolynomial $P_1(x)$ to the multiquadric Radial basis function (LMRBF) to improve the accuracy and constancy of the interpolation method.Moreover, a comparison between the LMRBF and the Cubic multiquadric Radial basis function (CMRBF) methodintroduced by the author [16] is presented.
To evaluate the accuracy of the LMRBF method we apply itinto two sets of scattered data points. The first set is taken from Franke [17] while the second set is taken from Loch [9]which is sampled from a real Frangipani and Anthurium leaf using a laser scanner. We found that the proposed method constructs accurate model of the leaf surface.Furthermore, the LMRBF produced slightly more accuarte leaf model than the CMRBF.

Keywords - Interpolation, virtual leaf, Radial basis function, AMS classification numbers.

I. INTRODUCTION

The application of surface fitting techniques to model the leaf surface is the main goal of the research given in this article, for instant, the application ofwater droplet or pesticide spreading on the leaf surface and modelling virtual plant, for more details see[1,11,18].Recentaly, many researchers have been interest in modelling leaves surfacessuch asKempthorne et al. [7]applied the discrete smoothingD^2-splines to model the cotton and wheat leaf.Oqielat et al. [20]introduced model for two types of leavesusing Hardy's multiquadrics RBF interpolation. Additionally, Oqielat[15,19]proposed a new hybrid clough-Tocher (CT)and radial basis function (RBF)method for modelling leaves surfaces.Furthermore, oqielat[21] compairedmany surface fitting techniques to reconstruct the surfaces of leaves and conclude that hybrid RBF-CT method produces more accuracte leaf model. More models introduced by [8,17,9]. In this paper, we implement the interpolant RBF with linear polynomial to reconstruct the leaf surface from 3D scattered points.

This paper composed of four sections. In § 2, an outline ofthe radial basis functionwith polynomial term and its shaped parameter isintroduced. Moreover, a numerical investigation to evaluate the accuracy of the LMRBF technique is given.In § 3, the LMRBF method is applied to reconstruct the surface ofFrangipani and Anthurium leaf, see figure 1. Furthermore, thechoice of the orthogonal distance regression plane as a reference plane for the leaf data is explained. Finally, a comparision betweenLMRBF and CMRBFmethod is deliberated.In § 4 future work and conclusion is presented.



(a)

(b)

Figure 1: Photos of the (a) Frangipani and (b) Anthurium leaf.

II. RADIAL BASIS FUNCTION APPROXIMATION WITH POLYNOMIAL REPREDUCTION

A radial basis function(RBF) method was originally presented by Hardy[10] and it is an efficient technique for resolving problems in many fields. The RBF method is based on calculating the distance between two points in d-dimensional. In order to obtain a smooth surface representation to evaluate the function values at points other than data points, radial basis function schemes have found applications in many areasin engineering and sciences and the solution of parabolic, elliptic and hyperbolic partial differential equations [4,5]. More information, introduction and reviews the theory of RBF is given in [22].

An examples of RBFs are Gaussian RBF $\phi(r) = e^{-(\alpha r)^2}$, thin plate splines $\phi(r) = r^2 \log(r)$, and Hardy's multiquadric $\emptyset(r) = \sqrt{r + \alpha^2}$ [4] which is adapted in this paper. The accuracy of interpolating scattered data with RBF depends on the parameter α [4,11]. Majdisova [13,14]prposed RBF interpolation approach of large data points that uses data structur for spare matrix storage and symmetry of matrix where the shaped parameter was determined experimentally.Rippa [25]suggested an algorithm for the optimal valueof the RBF parameter based on minimizes the cost function[23] and he showed that the value of α has great impact on the quality of the RBF approximation.

The cost function is given by:Let the vector $H = (H_1, \ldots, H_N)^T$ with $H_s = f_s - F^s(x_s), \quad s = 1, \ldots, N,$ where Γ^s is the interpolant to the data set with the point (x_s, f_s) removed, so that:

$$F^s(x) = \sum_{i=1, i \neq s}^{N} a_i^s \Re\left(\|x - x_i\|\right). \quad (1)$$

Rippa showed that $E_s = \frac{a_s}{a_s^s}$, where a_s is as defined in equation (1) and a^s is the solution of $Aa^s = e^s$, where e^s is the s^{th} co

Finally, the cost function $C(c)$ is given by $C(c) = \|H(c)\|_1$, and

$$c_{opt} = \arg\min_{c \in \mathbb{R}} \|H(c)\|_1.$$

A refinement algorithm of Rippa's algorithm was presented by Scheuerer[26].Gregory [3]introduce an extensions of the Rippa method that can be applied for the solution of PDF of RBF pseudo-spectral methods.

In this paper we explain themeshless technique based on the scheme of RBF interpolation whichleads to a solution of a linear system of equations.The RBF's are consists of two types. The local RBFs and global RBF. Using local RBF to approximate the data leads to less computational costof a linear system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ because A is a sparse matrix. While, the global RBFs build a large dense matrix and then they requires high computational costs. More details are found in [13,1427,28,24].

The approximation radial basis function $F(\bar{x})$ to the function $z(\bar{x})$ for a given data points $\bar{x}_i = (x_i, y_i)^T$ is given by the interpolation function $F : D \subset \Re^2 \rightarrow \Re$ that satisfy

$$F(\bar{x}_j) = \sum_{i=1}^{N} \gamma_i \phi_{i,j}(r) + \sum_{k=1}^{q} \mathrm{b}_k P_k(\bar{x}_j) = z_j(\bar{x}_j)$$

$; j = 1, \ldots, N \quad (3)$

where $\phi_{i,j}(r) = \sqrt{r + \alpha^2}$ is the multiquadric RBF, $r_i = \|x_j - x_i\|_2$ is the distance between the interpolation point $\bar{x}_j$ and the rest of the data points $\bar{x}_i, z_j(\bar{x}_j) = F(\bar{x}_j), j = 1, \ldots, N$, N is the number of nodes, $P_k(\bar{x}_j)$ is the polynomial of degree (k) in the coordinates space $\bar{x}$, q is the number of basis function for the polynomial, γ_i and b_k are the interpolation coefficients. In this paper, we added a polynomial of degree one i.e. k=1 to the multiquadric RBF to improve the stability of the overall method where the linear polynomial is given by:

$\mathrm{P}_1(\bar{\mathrm{x}}) = \mathrm{b}_x \mathrm{x} + \mathrm{b}_y \mathrm{y} + \mathrm{b}_0 = \bar{\mathrm{b}}^{\mathrm{T}} \bar{\mathrm{x}}$ where $\bar{x} = [x, y, 1]^T$.

Substitution $\mathrm{P}_1(\bar{\mathrm{x}})$ in Formula (3)gives:

$$F(\bar{x}_j) = \sum_{i=1}^{N} \gamma_i \phi_{i,j}(r) + \sum_{k=1}^{1} \bar{\mathrm{b}} P_1(\bar{x}_j) = \sum_{i=1}^{N} \gamma_i \phi_{i,j} + \bar{\mathrm{b}}^{\mathrm{T}} \bar{x}_j \quad (4)$$

and for conditionally positive definite RBF's, additional condition $\sum_{j=1}^{N} \gamma_j P_k(\bar{x}_j) = \bar{0}$ is applied.

The coefficients γ_i and $\bar{\mathrm{b}}^{\mathrm{T}}$ in formula (4) can be evaluated by enforcing the formula to be satisfied at the N data points about a point $\bar{x}$. Now, rewrite formula (4) in matrix form as follows:

$$\boldsymbol{Ax} = \boldsymbol{c} \quad (5)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \phi_{1,1} & \cdots & \phi_{1,N} & x_1 & y_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \phi_{N,1} & \cdots & \phi_{N,N} & x_N & y_N & 1 \\ x_1 & \cdots & x_N & 0 & 0 & 0 \\ y_1 & \cdots & y_N & 0 & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_N \\ b_x \\ b_y \\ b_0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rewrite the system $\boldsymbol{Ax} = \boldsymbol{c}$ as $\begin{bmatrix} \boldsymbol{M} & \boldsymbol{P} \\ \boldsymbol{P^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \bar{\gamma} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In this research, we implemented the truncated singular value decomposition (TSVD) of A to solve the system $\boldsymbol{Ax} = \boldsymbol{c}$, where the smallest singular value are ignored if they were less than or equal to the product of the ε (machine epsilon) with the largest singular value[12].

2.1 Numerical Experiment for the Franke Data

The results ofnumerical investigation for applying the LMRBF locally and globally is presented in this section. The root mean square error (RMS) given in the equation (6) is used to evaluate the accuracy of the LMRBF method for two sets of data points (consists of 100 point and 33 point) see figure 2,and six test functionstaken from[2]

$$\mathrm{RMS}=\sqrt{\frac{\sum_{k=1}^{m}[F(x_k)-z_k]^2}{m}} \quad (6)$$

where $F(a_i,b_i), i=1,2,\ldots,m$ are the interpolant LMRBF values at the data points (m) and z_i are the given function values at the same data points.

The global MRBF with linear polynomial interpolant (global LMRBF) is built from the $N=100$ points and then we used it to approximate the 33 points surface values, see figure 2. Furthermore, we build **the local MRBF with linear polynomial interpolant** (local LMRBF) "from the nearest 20 points ($n=20\subset N=100$)" about each point (p) of the 33 points, then the local LMRBF are used to estimate the surface value for that spacific point (p). In both cases we used Rippa algorithm [25] to estimate the RBF shape parameter c globally using the $N=100$ points.

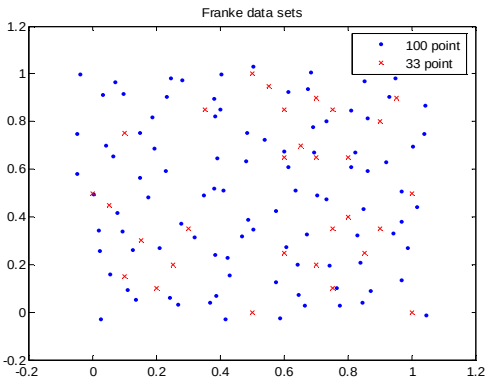


Figure 2: TheFranke data sets. The blue dots are 100 point and the red dots are 33 point defined on the unit square $[0,1]^2$.

Function	C	with linear polynomial	
		Global MRBF	Local MRBF
F1	0.1183	6.0e-003	6.5e-003
F2	0.0727	4.6e-003	4.2e-003
F3	0.2170	3.5e-004	6.3e-004
F4	1.1365	4.0e-005	4.0e-005
F5	0.3043	1.3e-004	1.3e-003
F6	2.4773	1.2e-004	2.4e-004

Table 1: A comparison of the RMS error using the globaland local MRBF with linear polynomial.

2.2.1 Results and Discussion

The RMS errors obtained by applying the global LMRBF and the local LMRBF for the six test functions is given in Tables 1. Notes that usinglocal LMRBF is computationally costly than using globalLMRBF because a dense matrix of size 20×20(see equation 5) demands to be resolved at each time we apply the local method. Moreover, the value of the parameter (c) is variant between 0.1 and 2.5 depends on the used function, and this is confirms that this has big influence on the quality of the RBF approximation (RMS error). Finally, the local LMRBF produces less accurate RMS than using theglobal LMRBF.Therefore, we bring this conclusion to the next section and investigate the suitability of the global LMRBF method for a real leaf.

III. APPLICATION OF THE MULTIQUADRIC RBF TECHNIQUE TO A REAL LEAF DATA SET

The proposed multiquadric Radial basis function with linear polynomial method (LMRBF) is employed in this section to model real Anthurium and Frangipani leaf surfacegiven in figures3 and 4. The LMRBF technique can not beapplied directly to the scanned leaf data points. Therefore we have to take in to account the following three issues.Firstly,to be able to capture the leaf surface data it is enought to sample the external points of the leaf. So the Anthurium and Frangipani leaf data points were sampled using a laser scannerby Loch [9].The collected surface points were 3,388 and 17 boundary points for the frangipani leaf, and for the Anthurium leaf they were4,688 and 79 boundary points(see figures 3 and 4). Hanan [6]used a software called PointPicker to collect the boundary points.Secondally, the reference plane of the scanned leaf points does not coincide with the coordinate system of xy-planeconsequentlya new refernce plane is required(see figures 3 and 4). In this paper,we usedthe orthogonal distance regression plane as a reference plane for the leaf data and the process of choosing this plane will explain afterward in this section. Finally, the multiquadric RBFparameterneed to be compute carefully where in this research we adapted Rippa method[25] to compute the RBF parameter.

3.1 Leaf reference plane

To model the leaf surface using our proposed LMRBF method a new reference plane for the data isnecessary for many reasons. Firstly, there is

measurement error in the coordinates system. Secondly,the plane of the scanned leaf data points may not necessarily coincide with the xy-plane (see figures 3 and 4). Finally,the chance of vertical surfaces and multivalued of the data points returned by the scanner see oqielat[15]. To overcome these issues an orthogonal distance regression plane that minimizes the perpendicular distances to the plane is used as reference plane, and then we rotates the coordinate system to obtain thenew reference plane as xy -plane(see figure 3 and 4).

Given data points $\ddot{O}_i = (x_i, y_i, z_i)^T,\ i = 1,2,\dots,N$, use the orthogonal distance regression planeis given by: $\dot{\eta}(x, y) = ax + by + c.$ (7)

To fit the data, start with the distance from a point to a plane and then find $a, b\ and$ c that minimizes $\dot{\eta}(a, b, c) = \sum_{i=1}^{N} \frac{|ax_i+by_i+c|^2}{a^2+b^2}$as follow:

Set the partial derivative with respect to c to be zero,

$\Rightarrow c = -\left(a\frac{\sum_{i=1}^{N} x_i}{N} + b\frac{\sum_{i=1}^{N} y_i}{N}\right) = -(ax_0 + by_0)$

Substitutes c into equation (7)we obtain$a(x - x_0) + b(y - y_0) = 0.$

Rewrite$(\dot{\eta})$ as:

$$\dot{\eta}(a, b) = \sum_{i=1}^{N} \frac{|a(x_i-x_0)+b(y_i-y_0)|^2}{a^2+b^2} \quad (8)$$

The data points after they are projected to the new reference plane (see figures3 and 4), are given by

$$\ddot{O}_i^{'} = \mathbb{Q}(x_i, y_i, z_i)^T i = 1,2,\dots,N$$

Where $\mathbb{Q} = \mathbb{Q}_x.\mathbb{Q}_y$act for the rotation matrix that rotates the unit normal vector of the orthogonal distance regression plane about the x-axis and y-axis, for more details [19].

3.2 Numerical Experiments for the Leaf Surface

This section reveals the outcomes of implementation the LMRBF interpolation technique to reconstruct the surface of the Anthurium and Frangipani leaf (shown in figure6), and then the result of the comparsion between our proposed method and the CMRBF method puplished by the author [16] is given also. One global LMRBF is build from 141 points (a subset of the 3405) and then used to construct the surface of the Frangipani leaf, while to construct the surface of the Anthurium leaf the global LMRBF is build from 212 points (a subset of the 4,767), see figure (5). The process of selection of the 212 points and 141 points are explained in (see oqielat et al. 2009) [41]. The 3264 Frangipani points and the 4,555 Anthurium leaf points remains, after the global LMRBF is build, are then used to assess the quality of the approximation of the LMRBF method. The root mean square error RMS (given in equation 6) and the maximum error given in equation (9) were used to compute the error.

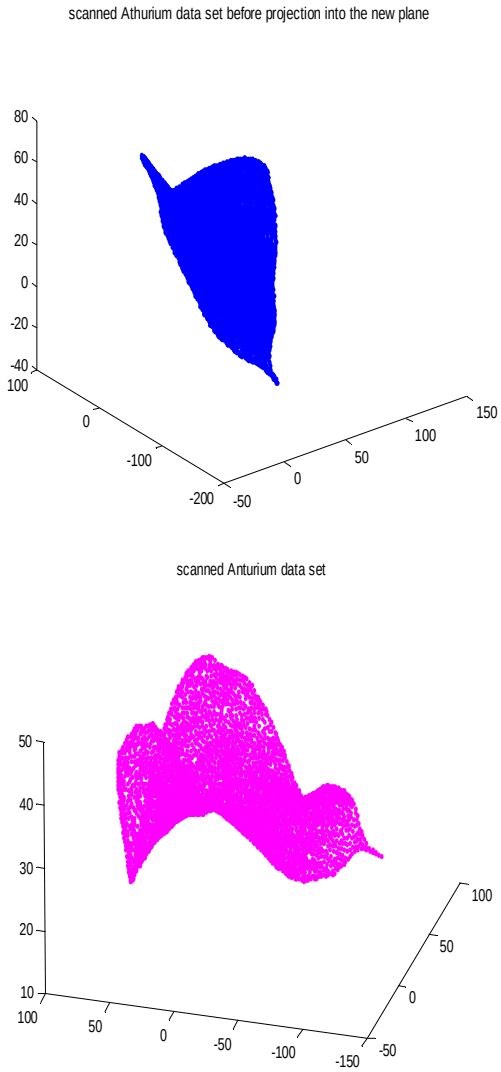
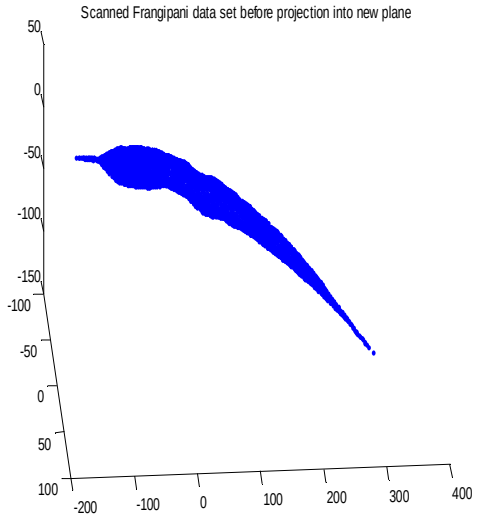


Figure 3: The 4,688scanned points for the Anthurium leaf data before projection into a new reference plane (left figure) and after projection (right figure).



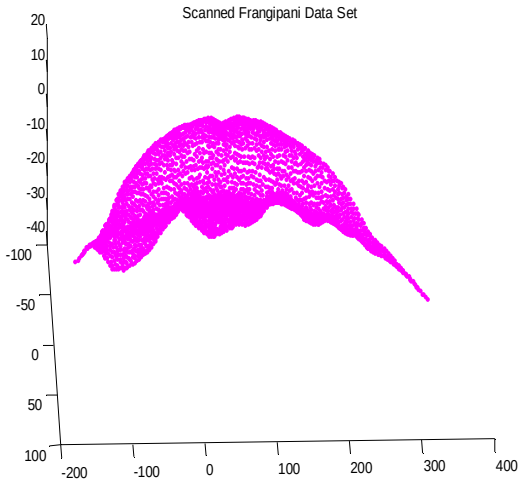


Figure 4: The 3,388 scanned points for the Frangipani leaf data before projection into a new reference plane (left figure) and after projection (right figure).

$$\text{Maximum Error} = \frac{\max\left(\left|F(a_i,b_i)-z_i\right|\right)}{\max(z_i)-\min(z_i)},$$

where $F(a_i,b_i), i=1,2,\ldots,m$ are the interpolant LMRBF values at the data points (m) and z_i are the given function values at the same data points.

Tables 2 and 3 show a comparison between the maximum errors (equation 9) and the RMS error (equation 6) for the Anthurium and Frangipani leaf data respectively using the Global LMRBF and global CMRBF method. In both methods the global MRBF is developed from 212 points for Anthurium leaf and 141 points for the Frangipani leaf. A total number of 4555 and 3264 data points were used to evaluate the accuracy of both methods. The MRBF parameter generated using Rippa algorithm and it is variant for both methods.

Methods	**With Linear Polynomial**	**With Cubic Polynomial**
	Global MRBF	**Global MRBF**
Maximum error	4.2e-002	4.7e00-2
Relative RMS	6.9e-003	7.8e00-3
Number of point tested	4555	4555
Number of Boundary points	79	79
The RBF shaped Parameter (c)	6.4	0.5

Table 2: A comparison of the RMS and maximum error for the Anthuriumleaf data points computed using the global LMRBF and the global CMRBF (introduced by the author [16].

Methods	**With Linear Polynomial**	**With Cubic Polynomial**
	Global MRBF	**Global MRBF**
Maximum error	8.7e-002	9.9e-002
Relative RMS	1.6e-002	1.7e-002
Number of point tested	3264	3264
Number of Boundary points	17	17
The RBF shaped Parameter (c)	0.5	0.1

Table 3: A comparison of RMS and maximum error for the Frangipani leaf data points computed using the global LMRBF and the global CMRBF (introduced by the author [16].

One observe from Table 2 that both methods generate slightly better RMS than maximum error. Moreover, the global LMRBF produces more accurate RMS and maximum error than the global CMRBF for both leaves. The tendencies described in Table 2 for the Anthurium leaf seems consistent with observations from Table 3 for the Frangipani leaf. In conclusion, Our proposed LMRBF method creates an accuarte representation of the leaf surface shown in figure 6, (b) and this is confirm our outcomes for Franke data set [2] given in section 3.

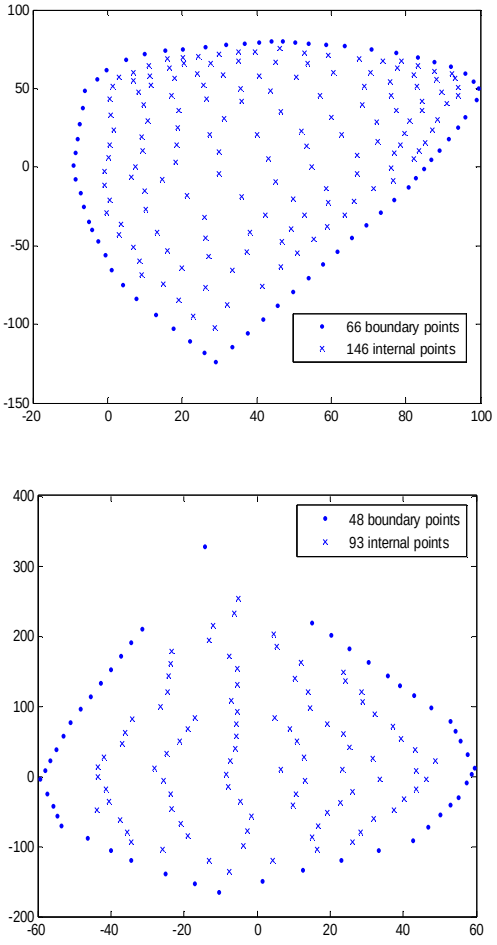


Figure 5: The212 points that used to construct a global MRBF for the Anthurium leaf (left figure). The figure on the right represent the 141 point that used to construct a global MRBF for the Frangipani leaf.

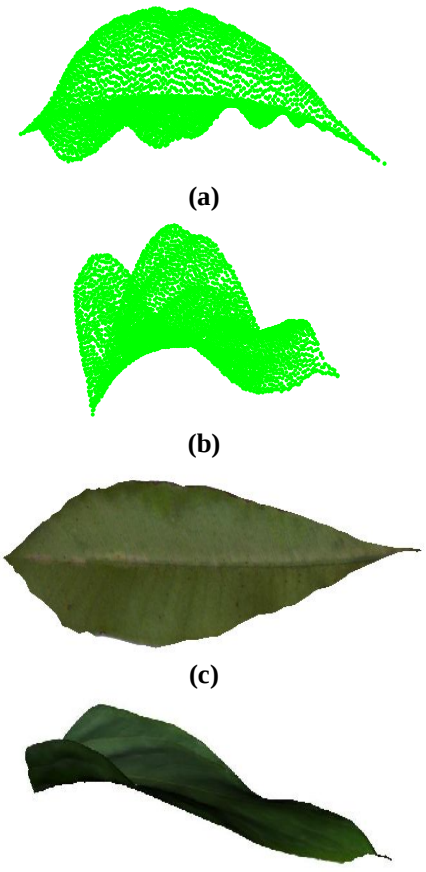


Figure 6: (a)& (b) The Frangipani & Anthuriumleaf surface model constructed from the points (shown in figure 3& 4) respectivelyusing the LMRBF method. (c) & (d) The corresponding visulization.

CONCLUSIONS AND FUTURE RESEARCH

In this paper we introduced a surface fitting method that combine linear polynomial with multiquadric RBF (LMRBF) to reconstruct the leaf surface from three-dimensional scanned data set. The proposed algorithm has been demonstrated to offer an accuarte model of the leaf surface (see figure 6. This model can be employed for modelling fluid movement such as spray droplet (pesticide or water) movement and spreading on a leaf surface and then will help in the treatments of different pesticide formulations. A comparsion of the LMRBF with the CMRBF method introduced by the author [16] is given and shown that the LMRBF produced more accurate representation of the leaf surface compared with the CMRBF method.

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