THE DIFFERENTIAL AND DELAY DIFFERENTIAL APPROACH IN THE ANALYSIS OF STABLE STATE EQUILIBRIUM PRICES USING CHARACTERISTIC EQUATION TECHNIQUES

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Abstract - The study compares the stability states of price adjustment differential models with and without delay parameter using roots of characteristic equations. The states of stability of both models were simulated using their particular solutions with inputs from same source. It was found that irrespective of initial prices set for the commodity, the current price for the differential models will always have the propensity to move monotonically to the equilibrium price defined for the system. On the other hand, the current price for the delay-differential models tends to oscillate and move away from the initial price, then with time, decreases and turns towards equilibrium, different from the system defined equilibrium price, due to the delay associated with the supply. It was deduced that whilst the equilibrium prices in the delay-differential models are not predictable due the time delay parameters associated with them, its counterpart without the delay are predictable, since differential models converge to the equilibrium price points defined for the system. Since most economic and natural phenomena are associated with delays, it is recommended that such systems are modeled using delay-differential equations to reflect realities of the models.

Keywords - Stability analysis, delay differential equations, differential equations and equilibrium prices.

INTRODUCTION

The oscillatory behavior of a delay differential equation and the associated differential equation without delay are not always the same [1]. In both theory and applications, the crucial difference is that a delay differential equation is infinite dimensional and it requires a bit more mathematical expediency as compared to their ordinary differential equations counterparts [2].

Time-delay dynamical systems are usually modeled by delay differential equations. Such delays often appear in many control systems either in the state, the control input, or the measurements, yet only few scientists apply these equations due to the complexity associated with their solutions [3]-[5].

Delay systems are infinite dimensional in nature and in many cases, time-delay is considered a source of instability. The issue of stability and the performance of time-delay control systems are therefore, both of theoretical and practical importance [2],[5].

The mathematical description of delay dynamical systems will naturally involve the delay parameter in some specified way. Time-delay models are abundant in nature. These models occur in a wide variety of physical, chemical, engineering, economic, weapons fire-control from moving platforms and biological systems because many processes include after effect phenomena in their inner dynamics [2],[6].

Recently, due to the understanding that most physical systems are time-delay inherent there has been increasing interest related to the theory of delay differential equations [7].
Characteristic roots (roots of the characteristic equation) usual provide qualitative information about variables whose evolutions are described by dynamic equations. Generally, if a differential equation is parameterized on time, the variable's evolution will be stable if and only if the real part of each root is negative. For difference equations, there is stability if and only if the modulus (absolute value) of each root is less than 1. For both types of equations, persistent fluctuations occur if there is at least one pair of complex roots [8].

Therefore, this study seeks to use delay differential equation that is a limiting case of these equations as price adjustment model and then qualitatively compare the state of stability with its differential counterpart using characteristic equation technique. This technique is applied in the study due to complexity of solutions for systems that are linearized around equilibria, especially when the solutions involve complex variables [8].

There is no doubt that the outcomes of the study would enhance the understanding of the qualitative behavior and solutions of delay differential and differential equations so that real-life systems that are time parameterized could be modeled using appropriate mathematical approach.

II. METHODS

This paper presents the dynamics of stability conditions of both differential and delay-differential

equations applied as price adjustment functions. Therefore, the two mathematical methods are used to model time dependent processes in economics, specifically, demand and supply of commodities and then qualitatively examine their stability behaviour using characteristic equation techniques.

2.1 Price Adjustment Differential Model without Delay

Given demand and supply functions of price with time equation respectively of the form

$$D_{p(t)} = \alpha + \beta P_{(t)}; \quad \beta < 0 \qquad (1)$$

$$S_{p(t)} = \lambda + \delta P_{(t)}; \quad \delta > 0 \qquad (2)$$

where α and λ are intercepts, β is the change in demand and δ is the change in supply.

The assumption is that the demand and supply of the commodity is dependent on only the price set for the commodity in a single market. Also, from the Walrasian assumption, the rate of change of price involving the (1) and (2) is given by

$$P'_{(t)} = \gamma(D_{p_{(t)}} - S_{p_{(t)}}). \qquad (3)$$

Substituting (1) and (2) into (3) gives

$$P'_{(t)} = \gamma(\alpha - \lambda + (\beta - \delta)P_{(t)}). \qquad (4)$$

Equation (4) is simplified if the following notations are adopted: let $-\rho = \gamma(\beta - \delta) > 0$ and

$p_e = \dfrac{\alpha - \lambda}{\delta - \beta} > 0,$ then dividing through (4) by (δ − β), we have

$$P'_{(t)} = -\rho(P_{(t)} - P_{(e)}), \qquad (5)$$

where P_e is the equilibrium price. At $P_t = P_e$ there is no price change. If $Z_{(t)} = P_{(t)} - P_e$ denotes the deviation from the $Z'_{(t)} = P'_{(t)}$ then (5) is reduced to the form

$$Z'_{(t)} = -\rho Z_{(t)}, \qquad (6)$$

which could be solved by method of characteristics if $Z_{(t)} = Ce^{\varphi t}$ where C is a constant and φ is the root of the (6). This implies that

$$\varphi Ce^{\varphi t} = -\rho Ce^{\varphi t} \qquad (7)$$

and thus,

$$(\varphi + \rho)Ce^{\varphi t} = 0 \Rightarrow \varphi = -\rho.$$

Using the notation under (5), the general solution in terms of price is given by

$$P_{(t)} = P_e + Ce^{-\rho t}. \qquad (8)$$

The particular solution at time $t = 0$, with initial condition $P_{(0)} = P_0$ is also obtained as

$$P_{(t)} = P_e + (P_0 - P_e)e^{-\rho t} \qquad (9)$$

This means that irrespective of the $P_{(0)}$ set for the (9), as $t \to \infty$, $e^{\infty} \to 0$ and so $P_{(t)} = P_e$ after a long period of time [9]..

2.2 Price Adjustment Differential Model with Delay

If (2) is considered to respond to price change with certain time-delay then supply functions of price with time-delay equation is given in the form

$$S_{p(t)} = \lambda + \delta P_{(t-\tau)}; \quad \delta > 0, \qquad (10)$$

where τ is the time needed for change in supply, in response to a unit change in price of the commodity. Then from (3), the rate of change is mathematically expressed as follows

$$P'_{(t)} = \gamma(\alpha - \lambda + \beta P_{(t)} - \delta P_{(t-\tau)}). \qquad (11)$$

If $q = \gamma(\alpha - \lambda)$ with $\alpha > \lambda$, $r = -\gamma\beta$ and $s = \gamma\delta$ then

$$P'_{(t)} = q - rP_{(t)} - sP_{(t-\tau)}. \qquad (12)$$

Using the transformation $Z_{(t)} = r(P_{(t)} - P_e)$, where P_e is the equilibrium price (same as used) in (4), then diving (12) through by $(\delta - \beta)$ gives

$$P'_{(t)} + r^2(P_{(t)} - P_e) + sr(P_{(t-\tau)} - P_e) = 0$$

and

$$Z'_{(t)} + rZ_{(t)} + sZ_{(t-\tau)} = 0. \qquad (13)$$

Equation (13) could be solved using characteristic techniques such that if $Z_{(t)} = Ce^{\mu t}$ then

$$\mu Ce^{\mu t} + rCe^{\mu t} + sCe^{\mu(t-\tau)} = 0. \qquad (14)$$

This implies that

$$(\mu + r)Ce^{\mu t} = -sCe^{\mu(t-\tau)}$$

and

$$(\mu + r) = -se^{-\mu\tau}.$$

By setting $m = \mu + r$, we have

$$me^{(m-r)\tau} = -s.$$

By multiplying through by $e^{r\tau}$, we obtain

$$me^{m\tau} = -se^{r\tau} \qquad (15)$$

Since s, r and τ are provided, the right hand side of the (15) could be represented by σ so that

$$F_{(m)} = me^{m\tau} + \sigma. \qquad (16)$$

It is obvious that no real and positive values of m satisfy this transcendental equation, however, for some range of σ there are complex roots with positive real parts. In this case, the solution would provide oscillations that increase exponentially.

Also, if $m = x \pm iy$, and $me^{m\tau} + \sigma = 0$, then

$$(x + iy)e^{(x+iy)\tau} + \sigma = 0$$

For which

$$(x + iy)e^{iy\tau} = -\sigma e^{-x\tau}$$

or

$$(x+iy)(\cos(y\tau)+i\sin(y\tau))=-\sigma e^{-x\tau}.$$

Then the following equations are obtained:

$$x\cos(y\tau)-y\sin(y\tau)=-\sigma e^{-x\tau} \quad (17)$$

and

$$y\cos(y\tau)+x\sin(y\tau)=0 \quad (18)$$

From (18), $x=-y\cot(y\tau), \quad y\neq 0$

We notice that

$$\lim_{y\to 0}-y\cot(y\tau)=\lim_{y\to 0}\frac{-y\tau\cos(y\tau)}{\tau\sin(y\tau)}$$

$$=\lim_{y\to 0}\left(\frac{-\cos(y\tau)}{\tau}\right)\lim_{y\to 0}\left(\frac{y\tau}{\sin(y\tau)}\right)$$

$$=\frac{-1}{\tau}.$$

Now, if $x\to\frac{-1}{\tau}, as, y\to 0$ and (17) and (18) are satisfied by $(x,y)=\left(\frac{-1}{\tau},0\right)$ then $\sigma=\frac{1}{\tau e}$.

Therefore, a particular solution of (13) in terms of price at $t=\tau=0$ is given by

$$P_{(t)}=2P_{(e)}+(P_{(0)}-P_{(e)})e^{-\mu t}+(P_{(0)}-P_{(e)})e^{-\mu(t-\tau)}, \quad (19)$$

where $\mu=(\frac{1+\tau r}{\tau})$ and (19) will also move to equilibrium irrespective of the initial price of the commodity. However, the movement would be controlled by the delay parameter in the system [9],[10].

III. RESULTS AND DISCUSSION

The results of the paper are now presented using same parameter values for both the differential and the delay differential price adjustment models to simulate the equilibrium prices when the systems are in their stable states using MatLab.

3.1 Equilibrium Price Set between the Initial Prices

The equilibrium prices are fixed at GHS 0.60 and GHS 0.65, in between the initial prices of GHS 1.70 and GHS 0.20 respectively. The price adjustment models are numerically run for both delay differential and differential systems and we had the following results.

Delay-Differential Models:

The time-delays are set at 1:00 and 0.20 at different occasions and then defined the equilibrium prices at stated earlier. The delay differential model was run numerically and following two graphical results obtained.

In Figure 1, when the initial prices were fixed at GHS 0.20, GHS 1.70 and the equilibrium also at GHS 0.60 GHS 0.65 respectively, the current price started rising far above the initial prices then at point in time began decreasing to settle at its own equilibrium (around GHS 1.20 to GHS 1.40) state due to the delay incorporated in the model.

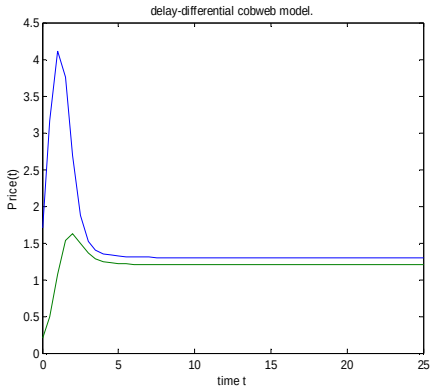

Figure 1: For delay of 1.00 and System Defined Equilibria of GHS 0.60 and GHS 0.65.

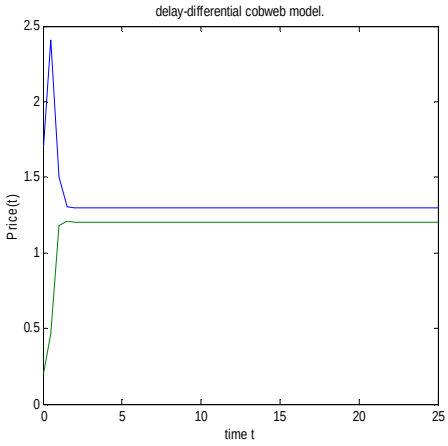

Figure 2: For a delay of 0.20 and System Defined Equilibria of GHS 0.60 and GHS 0.65.

From Figure 2, the current price initiated upward movement then suddenly turned, decreasing towards its own equilibrium price between GHS 1.20 to GHS 1.40 due to the time delay value of 0.02.

Differential Models:

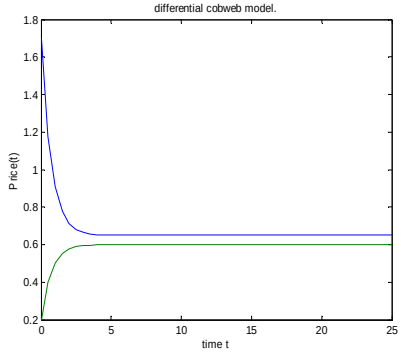

Figure 3: System Defined Equilibria of GHS 0.60 and GHS 0.65.

The differential model was also run numerically using the system defined equilibrium prices of GHS 0.60 and GHS 0.65 and we obtained results as appeared in Figure 3.

It is observed in Figure 3, that the current price moved monotonically to the system defined equilibrium prices of GHS 0.65 and GHS 0.60 from the initial prices of GHS 1.70 and GHS 0.20 respectively.

3.2 Equilibrium Price Set beyond the Initial Prices

The equilibrium price is now fixed above the initial prices at GHS 23.0, whilst the initial prices of the commodity maintained at GHS 0.20 and GHS 1.70. The system was run numerically and the following graphical representations obtained for both delay differential and differential models.

Delay-Differential Models:

The parameter values are run for the delay-differential price adjustment model with differential time-delay values but at same system defined equilibrium prices GHS 2.30 and we had the results as shown in Figures 4 and 5.

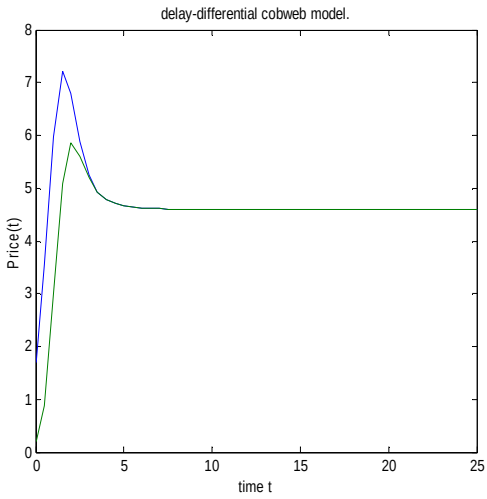

Figure 4: For a delay of 1.00 and System Defined Equilibrium of GHS 2.30.

In Figure 4, when only one equilibrium price was set at GHS 2.30, above the initial prices of GHS 0.20 and GHS 1.70, the oscillatory behavoiur of the delay price function did not change. At different initial prices, the current price moved upwards and then with time settled at equilibrium price of around GHS 4.60 which is about twice of the equilibrium price defined for the system.

In Figure 5, it is also observed that at common defined equilibrium price of GHS 2.30 and time-delay of 0.2, the current price converged to a new equilibrium point of GHS 4.60, when the system was fixed with initial prices of GHS 0.20 and GHS1.70.

Differential Models:

For common system defined equilibrium price of GHS 2.30, the numerical results of the differential price adjustment model also provided the results as follows.

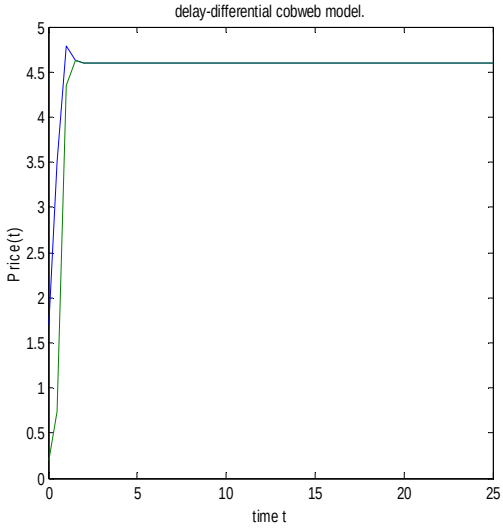

Figure 5: For a delay of 0.2 and System Defined Equilibrium of GHS 2.30.

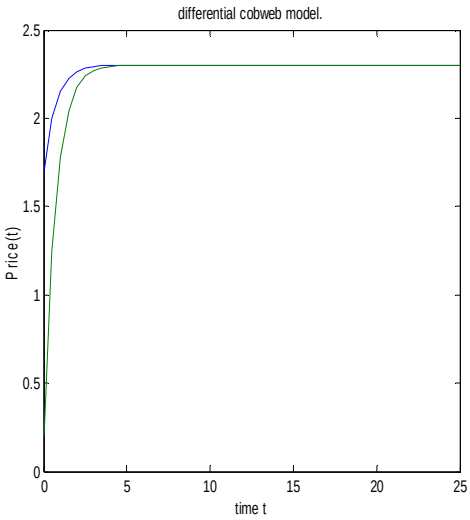

Figure 6: System Defined Equilibria of GHS 2.30.

From Figure 6, it is deduced that whenever the defined equilibrium price point of differential model is set beyond the initial prices, the current price will also move towards it.

CONCLUSION

This paper studies the dynamics of stability states of price adjustment differential equations with and without delay parameter using characteristic equation techniques. The states of stability of the two models were simulated using their particular solutions with inputs from same source.

It was found that irrespective of initial prices set for the commodity, the current price for the differential models will always have the propensity to move monotonically to the equilibrium price defined for the system.

On the other hand, the current price for the delay-differential models tends to oscillate and move away from the initial price, then with time decreases and turns towards to equilibrium which is different from the system defined equilibrium price due to the delay associated with the supply.

The equilibrium prices in the delay-differential models are not predictable due the time-delay parameters associated with them. However, the equilibrium price of differential models without delay could be predicted as such models converge to the equilibrium price points defined to be operated by the system.

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