AN EFFICIENT APPROACH FOR SOLVING FISHER'S EQUATION USING RADIAL BASIS-COLLOCATION TECHNIQUE

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Abstract: In this paper, we propose an efficient modification of the Collocationmethod which uses a radial basis function (RBF) as a basis function to findan approximate solution of a boundary value problem. We use the Fisherequation as a case study to examine the validity and applicability of theproposed technique. The numerical results show that the proposedmodification is efficient, applicable, easy to use, and more accurate thansome existing methods in the literature.

Keywords: Collocation Method, Radial Basis, Differential Equations, Fisher Equation.

I. INTRODUCTION

Boundary value problems (BVP's) are used in modeling a wide range of scienceand engineering problems. However, for many BVP's there are no knownanalytic solutions and even when it does exist it might be very difficult toobtain. In such cases, numerical approximations are frequently used. Adesirable attribute of these numerical techniques is to produce a goodapproximate solution while at the same time having a low to moderatecomputational complexity [1]. For example consider thesimplest reaction-diffusion equation concerning the concentration $u(x,t)$ which denotes the density of an advantageous of a single substance inone spatial dimension,

$$u_t = Du_{xx} + R(u) \quad (1.1)$$

If the reaction term vanishes, then the equation represents pure diffusionprocess and if thermal diffusivity term will come instead of diffusion term$D$ then equation will convert into a parabolic partial differential equationin one-dimensional in space. The choice $R(u) = \beta(1-u)$, where $\beta > 0$ in (1.1) yields the following Fisher equation that is used to describe the kineticadvancing rate of the advantageous gene also the spreading of biologicalpopulations [2, 3]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta u(1-u)$$

$$u(x,0) = u_0 \ (x),\ a \le x \le b \qquad (1.2)$$

$$u(a,t) = g_0(t), \quad u(b,t) = g_1(t),\ t \ge 0$$

The most popular techniques to solve BVP's are the finite difference, finiteelements and the shooting methods. Recently many researchers have beeninterested in solving BVP's using the collocation method [4, 5, 6, 7, 8, 9].

The attributes of Fisher's equation have been investigated theoretically byseveral authors. The study of traveling wave solution of Fisher's equationhas been analyzed by various computational methods. In [10], Fisher's equation was for the firsttime solved by RBF collocation method where the author's discretized timederivative of the equation by using the usual finite difference formula andspace derivative by the Crank-Nicholson formula between successive two-timelevels n and n+1. Wazwaz and Gorguis in [2] hadfound an approximate solutions to Fisher's equation and to a nonlineardiffusion equation of the Fisher model by applying the Adomian decompositionmethod, that has been commonly used in applied sciences, which is able ofmanipulating a universal type of diffusion problems. The method presents thesolution in a fast convergent series with elements that are elegantlycomputed.

Fisher's equation was also solved by the homotopy analysis method (HAM)which is a semi-analytical procedure to solve nonlinear ordinary/partial differential equations. In [11] the author solved Fisher equation using the sinccollocation method which methods are now known as an effective numericallyfor problems whose solutions may have singularities, or infinite domains, orboundary layers [12]. It has frequently beenidentified as important tools for problems in applied physics and engineering [13].

II. MODIFICATION OF THE COLLOCATION METHOD

In this section, we propose an effective method for solving the Fisherdiffusion reaction equation based on using radial basis functions (RBF) asbasis. We convert the collocation problem into a convex quadraticoptimization problem which presents a lot of advantages such that it is easyto solve and has several solvers such as Matlab.

Throughout this section we decompose the time space dependency in $u(x, t)$ into two terms, the first one is the space related term that is includedusing the RBF function of space x only, the second term is the

timedependent term $\alpha_k(t)$ which will be explained later and it carriesall information related to time. This decoupling of space and time enablesus to simplify the formulation of the problem using the new formulation thatenable us to transform the collocation problem into a convex one.

Consider the Fisher equation :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta u(1-u)$$
$$u(x,0) = u_0(x),\ x_a \le x \le x_b \qquad (2.1)$$
$$u(x_a,t) = g_0(t),\quad u(b,t) = g_1(t),\ t \ge 0$$

We propose to approximate the exact solution $u(x,t)$ by $U(x,t)$, where

$$u(x,t) \cong U(x,t) = \sum_{k=1}^{K} \alpha_k(t)\phi(x-\eta_k),\ (2.2)$$

Where the points $\{\eta_k\}_{k=1}^{K}$ are the centers and $\phi(.)$ is a RBF. Notice that the argument of $\phi(.)$ is alongthe x-axis (i.e. space). The time coefficients $\{\alpha_k(t)\}_{k=1}^{K}$ are functions of time are used to incorporate the *t*variable into theapproximation $U(x,t)$. One can use different forms of $\alpha_k(t)$. Forexample, the linear function $\alpha_k(t) = c_{k,1}t + c_{k,2}$ can be used.However, this choice proves not to be useful in simplifying the solution of $u(x,t)$ because the approximation might break at certain instance of time.

In this work we propose using $\alpha_k(t) = a_k e^{bt}$, where a_k is a scaling parameter and b is a decay factor. Thus,thefunction $U(x,t)$ can now be written as:

$$U(x,t) = \sum_{k=1}^{K} a_k e^{bt}\phi(x-\eta_k) \qquad (2.3)$$

Using the above approximation, we can write every term of the Fisherequation as follows:

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{K} a_k b e^{bt}\phi(x-\eta_k)$$
$$\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{K} a_k e^{bt}\frac{\partial^2\phi(x-\eta_k)}{\partial x^2}$$
$$u^2 = \sum_{k_1=1}^{K}\sum_{k_2=1}^{K} a_{k_1}a_{k_2}e^{2bt}\phi(x-\eta_{k_1})\,\phi(x-\eta_{k_2})$$

(2.4)

One important note is that the derivative (e.g., second derivative) of anyRBF function is always defined. In particular, the second derivative whichis of interest in this article can be written as:

$$\frac{\partial^2\phi(x-\eta_k)}{\partial x^2} = g_k\phi(x-\eta_k),$$

where the term g_k depends on the choice of the RBF.It is now possible to write the Fisher equation in the following form:

$$\sum_{k=1}^{K} a_k[b e^{bt} - \alpha g_k - \beta]\phi(x-\eta_k) +$$
$$\sum_{k_1=1}^{K}\sum_{k_2=1}^{K}\beta\, a_{k_1}a_{k_2}e^{2bt}\phi(x-\eta_{k_1})\,\phi(x-\eta_{k_2}) = 0$$

(2.5)

The above equation can be evaluated at the different collocation points,which are the points $\{x_n\}_{n=1}^{N}$, thus resulting in *N*-equationsfor each time instant *t*. If we divide the time axis $t_0, t_1, ..., t_{m-1}$ then we have a set of *N* by*M* equations. Wecan write the set of these equations in compact form as follows

$$\mathbf{a^t Q\, a} = \mathbf{f^t a} \qquad (2.6)$$

where**a** is the vector of the coefficients (i.e.**a** = (a_1, a_2, …, a_k)^t) and **Q** is a quadratic matrix of size *K* by *K*, where the (k_1, k_2) entry is given by:

$$Q(k_1,k_2) = \frac{1}{2}\beta\sum_{n=1}^{N}\sum_{m=1}^{M}\phi(x_n-\eta_{k_1})\,\phi(x_n-\eta_{k_2})e^{bt_m}$$

(2.7)

andthe k^{th} entry of vector **f** is given by:

$$\mathbf{f}(k) = \sum_{n=1}^{N}\sum_{m=1}^{M}[be^{bt_m} - \alpha\, g(t_m - x_n) - \beta]\,\phi(x_n-\eta_k)$$

(2.8)

We note that the equation (2.6) is of a quadratic form. Our goal isto find the best values of the coefficients a_k that produce the bestapproximation $U(x,t)$. To achieve this, one can use the initialand boundary conditions of the Fisher problem. Having the initial condition that $u(x,0) = u_0(x)$, we can obtain

$$\mathbf{U_{init}} = \mathbf{u_0},\qquad (2.9)$$

where the (n,k)entry of $\mathbf{U_{init}}$ correspondsto the n^{th} point x_n and the k^{th} RBF centers η_k is given by

$$\mathbf{U_{init}}(n,k) = e^{bt_0}\,\phi(x_n-\eta_k)$$

Similarly, both the boundary conditions can be included. Define the matrix $\mathbf{U}_{x=x_a}$ in which the (m,k)entry corresponds to the m^{th}time instant and the k^{th} collocation point η_k and is given by:

$$\mathbf{U}_{x=x_a}(m,k) = e^{bt_m}\,\phi(a-\eta_k)$$

We also define the vector $g_0=(g_0(t_1), g_0(t_2),..., g_0(t_{m_a}))^t$, where m_a is the number of points on the*t*-axis. Therefore, the conditions on the boundary at point $x = x_a$ is given by:

$$\mathbf{U}_{\mathbf{x}=\mathbf{x}_a}\mathbf{a} = \mathbf{g_0} \qquad (2.10)$$

In similar fashion, we obtain the following boundary conditions at point $x = x_b$ using the same notation:

$$U_{x=x_b}(m,k) = e^{bt_m}\ \phi(b-\eta_k)$$

along with vector $g_1 = (g_1(t_1), g_1(t_2),..., g_1(t_{m_b}))^t$, where m_b is the number of points on the *t*-axis.This resultsgive us the following system of equations:

$$\mathbf{U}_{\mathbf{x}=\mathbf{x}_b}\mathbf{a} = \mathbf{g}_1 \quad (2.11)$$

Having the developed formula in (2.6) along with the conditionsin equations (2.9), (2.10), and (2.11), we can formulate theRBF collocation problem as a constrained optimization problem. Morespecifically, the optimization problem can be formulated as:

$$\underset{\mathbf{a}}{Min} \quad \mathbf{a^t Q a} = \mathbf{f^t a}$$

$$s.t. \begin{cases} \mathbf{U}_{\mathbf{init}} = \mathbf{u_0} \\ \mathbf{U}_{\mathbf{x}=\mathbf{x}_a}\mathbf{a} = \mathbf{g_0} \\ \mathbf{U}_{\mathbf{x}=\mathbf{x}_b}\mathbf{a} = \mathbf{g}_1 \end{cases}$$

We note that the above optimization problem can be easily solved. This isdue to the convexity of the cost function and the linearity of theconstraints. Several convex optimization solvers are available.

3. Implementation and Demonstration

In this section we consider some illustrative examples to demonstrate theperformance and efficiency of the proposed method for comparison reasons. InExample 1 we compare our proposed method with the Sinc collocation methodgiven in [11]. In Example 2 we compare our proposed method with the Adomaian method givenin [2] and the homotopy method given in [3].

Example 1:

Consider the following Fisher equation:

$$u_t = u_{xx} + (1-u), \quad -1 \le x \le 1, \quad t > 0$$

$$u(-1,t) = u(1,t) = 0$$

$$u(x,0) = 0$$

The exact solution of this equation is given by:

$$u(x,t) = 1 - \frac{\cosh(x)}{\cosh(1)} - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos((2n-1)\frac{\pi}{2}x)}{(2n-1)[(2n-1)^2\pi^2+4]}\exp\{-[1+(2n-1)^2\frac{\pi^2}{4}$$

Table1 shows the maximum error E_{max} resulting from theSinc-based approximation proposed in [7] as well as thatof our proposed quadratic formulation for different numbers *N* of Sinc andRBF functions. We note here that we used the exponential RBF with a shapeparameter *r*=10.

Table 1: Comparison between Sinc collocation and RBF collocation

N	$E_{max,\ sinc}$	$E_{max,\ RBF}$	$E_{Ave,\ RBF}$
8	4.139 x 10^{-3}	2.5094 x 10^{-5}	9.237 x 10^{-4}
16	8.661 x 10^{-4}	5.2766 x 10^{-5}	7.554 x 10^{-5}
24	4.106 x 10^{-4}	5.6805 x 10^{-5}	7.946 x 10^{-6}
32	2.237 x 10^{-4}	6.16 x10^{-5}	9.975 x 10^{-7}

We note that the average error for the proposed RBF method decreaseas we increase the number of the RBF centers. We also note that the proposedmethod out performs the sinc collocation method for most values of *N*.Next we investigate the effect of the shape parameter *r* on theapproximation error of the proposed method. In particular we examine themaximum and the average error over the interval $t \in [0, 0.1]$ with *N*=16 RBF functions.The results are tabulated in table 2.

Table 2: The effect of the shape parameter *r*

r	E_{max}	E_{Ave}
3	9.647 x 10^{-5}	3.317 x 10^{-6}
5	1.109 x 10^{-5}	3.629 x 10^{-6}
7	1.9375 x 10^{-5}	6.106 x 10^{-6}
10	2.351 x 10^{-4}	7.554 x 10^{-5}

We note on this case that as *r* increases from 3 to 10 both maximumand average errors increases as well. Obviously, thechoice of *r* gives a degree of freedom in obtaining the solution of the Fisher equation.

Example 2:

Consider the following Fisher equation

$$u_t = u_{xx} + u(1-u), \quad u(x,0) = \lambda$$

The exact solution is $u(x,t) = \frac{\lambda e^t}{1-\lambda+\lambda e^t}$

For the above example we set $\lambda = 0.1$.

Table 3 summarize the averageand the maximum errors between the exact and the approximate solutions fordifferent approximation orders for the Adomian and Homotopy methods. And table 4 shows both the maximum and average errors for the solution using the proposed method for different values of *r* and *N* =8 The maximum and the average errors for using the proposed method}

Table 3: Average and maximum errors of the Adomian and the Homotopy methods

Order	$E_{max,\ Adm,\ Hom}$	$E_{Ave,\ Adm,\ Hom}$
$u_0(x,\ t)$	9.036 x 10^{-4}	4.51 x 10^{-4}
$u_1(x,\ t)$	3.06 x 10^{-6}	1.322 x 10^{-6}
$u_2(x,\ t)$	6.89 x 10^{-9}	2.069 x 10^{-9}
$u_3(x,\ t)$	2.45 x 10^{-12}	6.394 x 10^{-13}
$u_4(x,\ t)$	5.4761 x 10^{-14}	1.29 x 10^{-14}

Table 4: Ave. and max. errors of the proposed method

r	$E_{max, RBF}$	$E_{Ave, RBF}$
0.1	9.01×10^{-16}	5.412×10^{-16}
0.5	2.345×10^{-15}	7.35×10^{-16}
1	1.7069×10^{-15}	7.077×10^{-16}
2	3.511×10^{-15}	3.511×10^{-15}
3	2.13×10^{-14}	2.706×10^{-15}

We note that the proposed quadratic optimization method provides avery accurate approximation of the above problem and is better than both the Adomian and the Homotopy methods up to the fourth approximation order.

CONCLUSION

In this paper we present a new modification for the RBF collocation method, to find an approximate solution for Fisher equation. We convert theresulting system into a convex quadratic optimization problem. Numerical experiments out performed to show the effectiveness of theproposed approach verses other methods found in the literature. Tables 1and 2 show that the average error for the proposed method decrease as we increasethe number of the RBF centers, and it out performs the sinc collocationmethod for most values of N. Tables 3 and 4 show that our proposed method provides a very accurate approximation and is better than both the Adomian and the Homotopy methods up to the fourth approximation order.

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