# UNSTEADY FLOW OF AN INCLINED SYMMETRIC THIN LIQUID FILMS

**JOSEPH G. ABDULAHAD**

Mathematics Department – Faculty of Science-Zakho University - Iraq
E-mail: dr.joseph50 @ yahoo.com, alshamanijoseph@gmail.com

**Abstract -** In this paper we construct a mathematical model to describe the unsteady incompressible flowwith in an inclined symmetric double sided thin liquid films in two dimensions. We apply the Navier – Stokes equations with an appropriate boundary conditions to obtain the non – linear differential equation of the film thickness.

**Keywords -** Thin Liquid Films, Navier-Stokes Equations, Dimensional Analysis.

## I. INTRODUCTION

The nature of physical surfaces has been studied over a long period of time. The macroscopic and molecular properties of the surface were investigated by physical, chemical and biological scientists, while the mathematicians have been concerned with mathematical problems, especially with the solution of the boundary value problems which arise from a macroscopic representation of the molecular properties of the surface. The mechanics of free surface liquid film is important in many applications. [1] considered the steady flow of a horizontal symmetric thin liquid film, the differential equations that governs such flow is derived and solved numerically. A mathematical model was considered by [2] to describe the unsteady flow in a horizontal thin liquid films and conclude that the film thickness of some liquids namely glycerin and olive oil decreases as the time increases. [3]investigated the nature of the boundary conditions at the free surface of the liquid, when the liquid is in the form of a moving thin film and developed a theory of steady motion in a single semi–infinite film with zero shear-stress surfaces rather than the no slip condition as Mysel s[7] suggested that the surface of the film behaves as a rigid surface.. A mathematical model is constructed by [4] to describe the thinning process of an inclined thin liquid film over a solid surface that drainages under the action of gravity. The stability analysis of thin liquid films are investigated by [5]by using the long wave method. [6] was concerned with the dynamics of an inclined thin liquid films of variable thickness for steady and unsteady flow and when the film is stationary and uniform.

The purpose of this work is to study the unsteady flow of an inclined symmetric double-sided thin liquid film. The governing non-linear differential equation is obtained analytically and the solution for the thickness of the liquid film equation is determined numerically. The similarity method is used to isolate the time dependence and then the shape of the film will depend on one variable only.

## II. FORMULATION AND GOVERNING

To describe the flow of a viscous fluid within a double sided inclined symmetric thin film in two dimensions, we take the Cartesian coordinates x and z in which the x axis is the axis of symmetry, and the flow is predominantly in the x directions and the z axis is perpendicular to the plane of the film. In figure .1.the transverse dimensions are greatly enlarged by comparison with the longitudinal dimensions for clarity of presentation

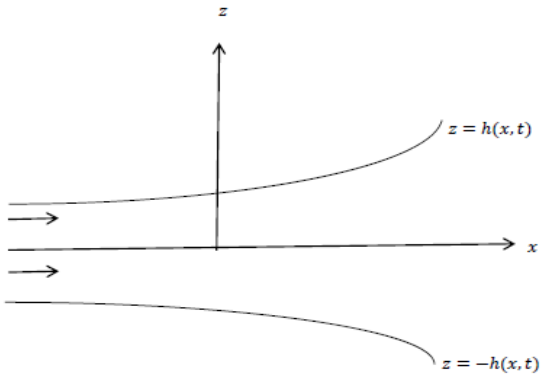



**Fig.1. Physical model for symmetric inclined thin liquid film**

Let the equations of the bounding surfaces of the film be

$$z = \pm h(x,t) \qquad (2.1)$$

And $\frac{\partial h}{\partial x} << 1$ for all x and t.

In differential form, the Navier-Stokes equations in x and z directions respectively are

$$\rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}] = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] + \rho g sin\theta \qquad (2.2)$$

$$\rho[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}] = -\frac{\partial p}{\partial y} + \mu[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}] + \rho g sin\theta \qquad (2.3)$$

Let the velocity distribution be defined as

$$u = u(x,t)$$

$v = 0$ (2.4)

$$w = -z\frac{\partial u(x,t)}{\partial x}$$

And the pressure is a function of x and t only.

The distribution (2.4) satisfies the continuity equation for incompressible flow which has the form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (2.5)$$

Since $z = h(x,t)$is the equation of the free surface of the liquid film, then the conservation of mass across the film thickness has the form:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \qquad (2.6)$$

Since the slop of the liquid film is so small, the distribution satisfies the following boundary conditions:

The tangential stress condition

$$\tau = \mu(\frac{\partial u}{\partial z})_s = 0 \qquad (2.7)$$

Where the subscript s denotes values at the bounding surface of the liquid film.

And the normal stress condition at the surface is given by

$$\omega = (-p + 2\mu\frac{\partial u}{\partial z})_s \qquad (2.8)$$

From the continuity equation (2.5), equation (2.8) gives

$$\omega = (-p - 2\mu\frac{\partial u}{\partial x})_s \qquad (2.9)$$

Where $\sigma$ is the surface tension of the liquid film and since the slop of the film is so small the curvature k of the film reduces to give

$$k = \frac{\partial^2 h}{\partial x^2} \qquad (2.10)$$

Also, on the surface of the film , the normal component of stress is given by

$$\omega = \sigma\frac{\partial^2 h}{\partial x^2} \qquad (2.11)$$

Now from equations (2.9) and (2.11) and on the surface of the film, we get

$$p = -2\mu\frac{\partial u}{\partial x} - \sigma\frac{\partial^2 h}{\partial x^2} \qquad (2.12)$$

Equation (2.12) is valid everywhere.

Differentiate equation (2.12) with respect to x, we have

$$\frac{\partial p}{\partial x} = -2\mu\frac{\partial^2 u}{\partial x^2} - \sigma\frac{\partial^3 h}{\partial x^3} \qquad (2.13)$$

From (2.4), the longitudinal equation of motion in x direction (2.2) reduces to give

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \rho g\sin\theta \qquad (2.14)$$

From equations (2.13) and (2.14), we have

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = 3\mu\frac{\partial^2 u}{\partial x^2} + \sigma\frac{\partial^3 h}{\partial x^3} + \rho g\sin\theta \qquad (2.15)$$

Again from the transverse equation of motion (2.3) and from the velocity distribution (2.4), we obtain

$$\frac{\partial p}{\partial z} = \left[\rho u\frac{\partial^2 u}{\partial x\partial t} + \rho u\frac{\partial^2 u}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x}\right)^2 - \mu\frac{\partial^3 u}{\partial x^3} - \rho g\cos\theta\right]z \qquad (2.16)$$

Integrating both sides of equation (2.16) with respect to z, we get

$$p = \left[\rho u\frac{\partial^2 u}{\partial x\partial t} + \rho u\frac{\partial^2 u}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x}\right)^2 - \mu\frac{\partial^3 u}{\partial x^3} - \rho g\cos\theta\right]\frac{z^2}{2} + f(x,t) \qquad (2.17)$$

That is

$$p = p(x,t) + O(z), |z| \le h \qquad (2.18)$$

Thus equations (2.6) and (2.15) are the governing equations of the inclined thin liquid films for unsteady flow.

## III. FLOW WITH NEGLIGIBLE INERTIA

We now consider unsteady flow within a liquid film but with negligible inertia, and thus the longitudinal equation of motion (2.15) reduces to give

$$3\mu\frac{\partial^2 u}{\partial x^2} + \sigma\frac{\partial^3 h}{\partial x^3} + \rho g\sin\theta = 0 \qquad (3.1)$$

Integrating both sides of equation (3.1) twice with respect to x, we get

$$u = \frac{1}{3\mu}\left(G(t)x + F(t) - \sigma\frac{\partial h}{\partial x} - (\rho g\sin\theta)\frac{x^2}{2}\right) \qquad (3.2)$$

Where G(t) and F(t) are arbitrary functions of t and can be determined from asymptotic or initial conditions.

Case1: If $G(t) \neq 0$, we write$H(t)x = G(t)x + F(t)$. From equation (2.6), equation (3.2) after some simplifications becomes

$$3\mu\frac{\partial h}{\partial t} + H(t)x\frac{\partial h}{\partial x} - \frac{1}{2}\rho g\sin\theta x^2\frac{\partial h}{\partial x} - \sigma\left(\frac{\partial h}{\partial x}\right)^2 + H(t)h - \rho g\sin\theta xh - \sigma\frac{\partial^2 h}{\partial x^2}h = 0 \qquad (3.3)$$

Case2: If $G(t) = 0$, we have

$$3\mu\frac{\partial h}{\partial t} + F(t)\frac{\partial h}{\partial x} - \rho g sin\theta x^2\frac{\partial h}{\partial x} - \sigma\left(\frac{\partial h}{\partial x}\right)^2 - \rho g sin\theta x h - \sigma\frac{\partial^2 h}{\partial x^2}h = 0 \qquad (3.4)$$

## IV. SIMILARITY SOLUTION AND NON-DIMENSIONAL ANALYSIS

For case1, let us suppose that the length scale associated with x and h vary with t in a different way. For non-trivial solution of equation (3.3), we introduce the non -dimensional variables as follows:

$$x = \frac{\mu}{\rho g t}\varepsilon$$
$$h(x,t) = \frac{\mu^3}{\rho^2 g^2 \sigma t^3} f(\varepsilon)$$
$$H(t) = k_1\frac{\mu}{t}$$

Where $k_1$ is of dimension $\frac{cm\, sec^2}{gm}$

Thus equation (3.3) in non-dimensional form becomes

$$f(\varepsilon)\frac{\partial^2 f}{\partial\varepsilon^2} + \left(\frac{\partial f}{\partial\varepsilon}\right)^2 + \sin\theta\,\varepsilon f(\varepsilon) + \frac{1}{2}\sin\theta\,\varepsilon^2\frac{\partial f}{\partial\varepsilon} - 4\varepsilon\frac{\partial f}{\partial\varepsilon} + 8f(\varepsilon) = 0 \qquad (3.5)$$

For case2, when $G(t) = 0$, we define the following non-dimensional variables:

$$x = \frac{\mu}{\rho g t}\varepsilon$$
$$h(x,t) = \frac{\mu^3}{\rho^2 g^2 \sigma t^3} f(\varepsilon)$$
$$F(t) = k_2\frac{\mu^2}{\rho g t^2}$$

Where $k_2$ is of dimension $\frac{sec^2}{gm}$.

And thus equation (3.4) in dimensionless form has the form

$$f(\varepsilon)\frac{\partial^2 f}{\partial\varepsilon^2} + \left(\frac{\partial f}{\partial\varepsilon}\right)^2 + \sin\theta\,\varepsilon f(\varepsilon) + \frac{1}{2}\sin\theta\,\varepsilon^2\frac{\partial f}{\partial\varepsilon} - (3\varepsilon + 1) * \frac{\partial f}{\partial\varepsilon} + 9f(\varepsilon) = 0 \qquad (3.6)$$

It is to be noted that and without loss of generality we assume that $k_1 = k_2 = 1$

Let $f(\varepsilon) = \varepsilon^{-n}$, where $\varepsilon \to \infty$

Substituting $f(\varepsilon)$ and its derivative in equation (3.5), we get

$$n(n+1)\varepsilon^{-n-1} + n^2\varepsilon^{-n-1} + \sin\theta - \frac{1}{2}\sin\theta\, n + 4n\varepsilon^{-1} + 8\varepsilon^{-1} = 0 \qquad (3.7)$$

and then in equation (3.6)and by taking the $\lim_{\eta\to\infty}$, we have $n = 2$. That is the solutions of the equations for both cases are asymptotic to $\frac{1}{\varepsilon^2}$ as $\varepsilon \to \pm\infty$.

for case one some solution curves of equation (3.5) are shown in the following figures.

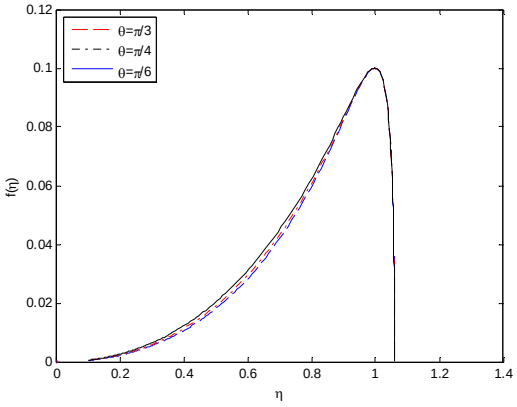
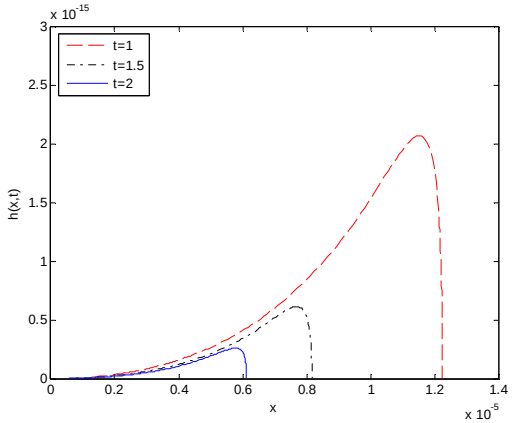
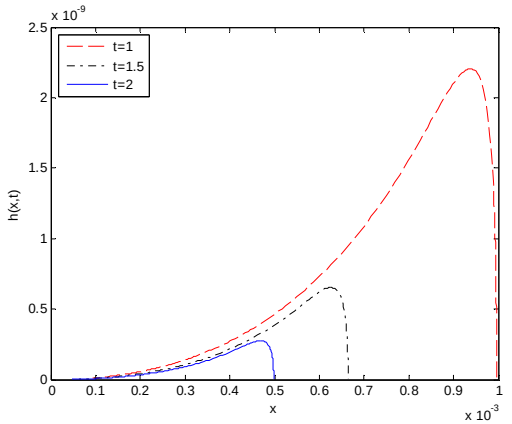


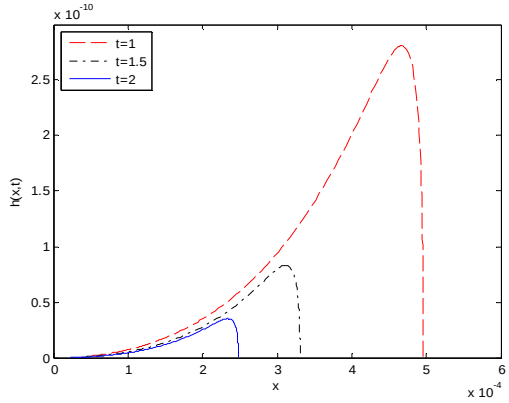

**Fig.2.The solution curve of equation (3.5) in $(\varepsilon, f(\varepsilon))$ plane.**



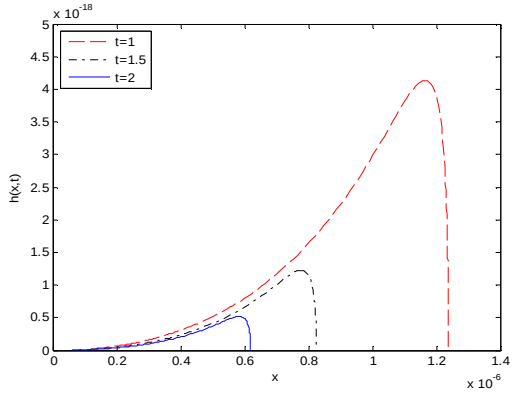

**Water**




**Olive oil**

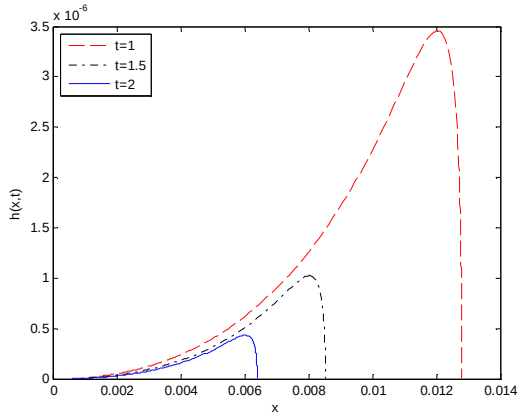



**Linseed oil**




**Mercury**

**Fig.3. Film thickness in (x, h(x, t)-plane of case one for water, olive oil, linseed oil, mercury and glycerin for t=1,1.5,2.**




**Glycerin**

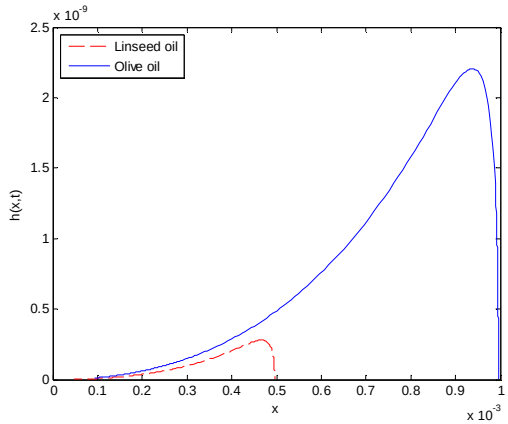



**Fig.4. Comparison between the film thickness of Olive oil and Linseed oil in case one.**

For case two some of the solution curves of equation (3.6) are presented in the following figures.

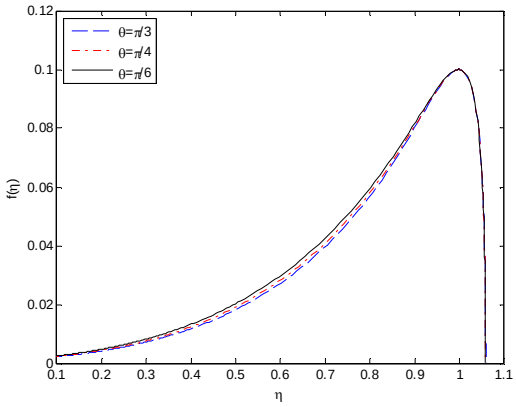
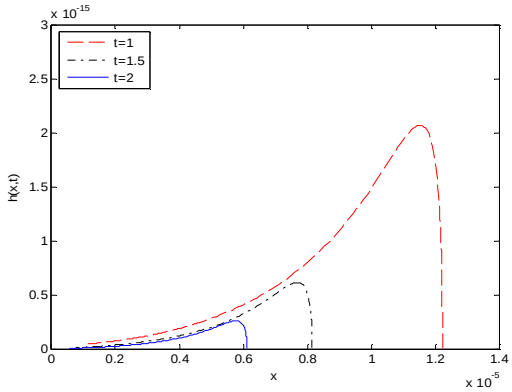
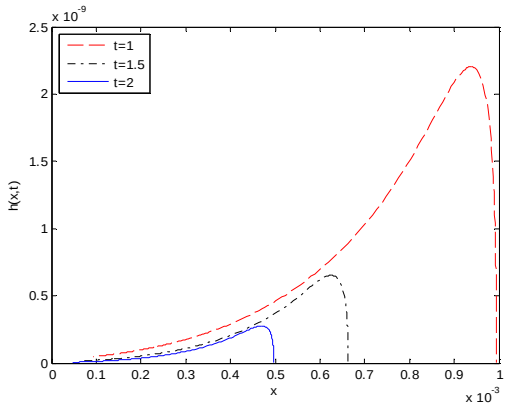


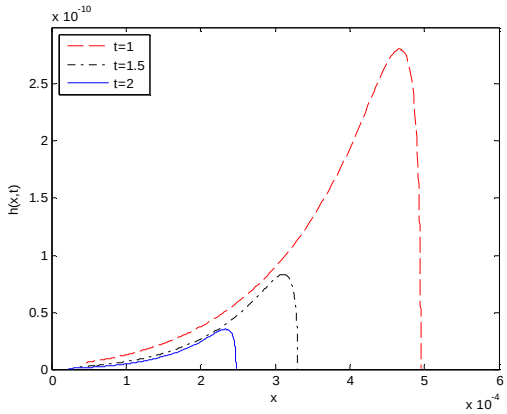

**Fig.5.The solution curves of equation (3.6) in $(\varepsilon, f(\varepsilon))$ plane**




**Water**




**Olive oil**




**Linseed oil**

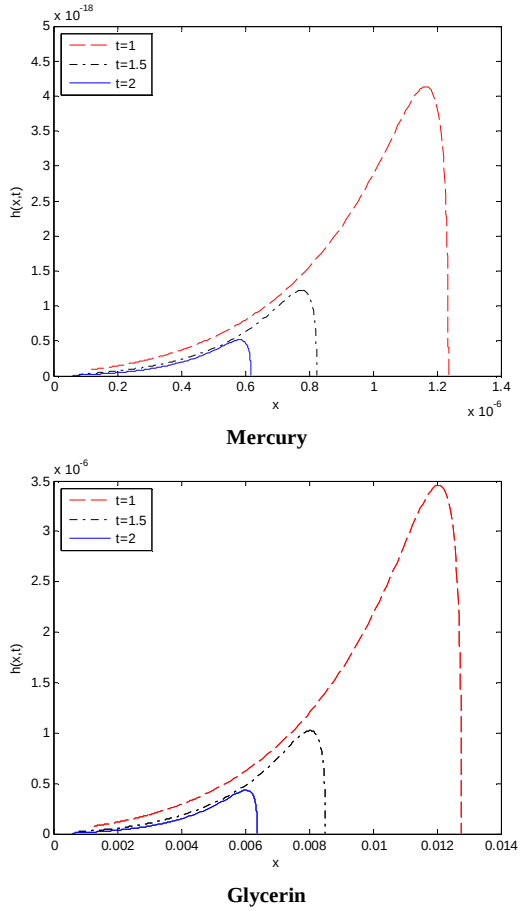



**Fig .6.Film thickness in (x, h(x, t)-planeof case two for water, olive oil, linseed oil, mercury and glycerin for t=1, 1.5, 2.**

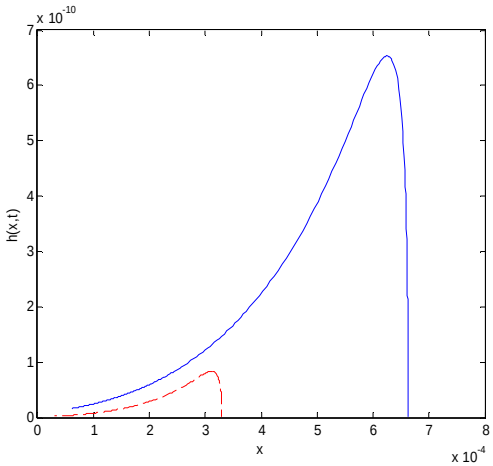



**Fig.7.Comparison between the film thickness of Olive oil and Linseed oil in case two.**

## CONCLUSIONS

For unsteady flow within double-sided inclined symmetric thin liquid films a non-linear differential equation that governs such flow is derived. We use the similarity method by which the thickness of the liquid film will depend on one variable $\varepsilon$ only. The solution of the governing equations of the liquid film thickness is obtained numerically. For case one, some of the solution curves in $(\varepsilon, f(\varepsilon))$-plane of equation (3.5) and for different values of the inclination angle $\theta$ are presented in fig.2.which shows that as the inclination angle increases the thickness of the film increases. For all liquids namely water, linseed oil, olive oil, mercury and glycerin, the film thickness decreases as the time increases and the reason for this may be the film extracts some of its liquids as the time increases, and shown in fig.3. in$(x, h(x,t))$ −plane. Fig..4. shows that the thickness of the linseed oil is less than that of the olive oil at a fixed value of time and this may be because the viscosity of the linseed oil is smaller than that of the olive oil. A similar conclusion for case two can be seen from the rest of the figures. Finally and from the macroscopic point of view we can conclude that this theory can be applicable for thin films of glycerin, marginally applicable for olive oil and linseed oil and can never be applicable for thin films of water and mercury.

## REFERENCES

[1] J. Abdulahad, S. Rasheed and H. AhmadI,“ Steady Flow of a Horizontal Double-Sided Symmetric Thin liquid Films”, IOSRJournal of Mathematics, vol. 3,pp.102-107,2016.

[2] J. Abdulahad,S. Rasheed and H. AhmadI, “Unsteady Flow in a Horizontal Double – Sided symmetric Thin Liquid Films”, INCAS BULLTIN, vol.9, pp. 3-16,2017.

[3] J.Abdulahad,, “ Liquid Surface Conditions”, Ph.D. thesis, School of Computer Science and Applied Mathematics”, University of Essex, England,1989.

[4] J. Abdulahadand S.Derbaz, “The Drainage of Thin Liquid Films on an Inclined Solid Surface”, IOSR Journal of Mathematics, vol.10, pp. 73-80,2014.

[5] J. Abdulahad and I. Hamad, “Stability Analysis of Thin Films By Long –Wave Method,IOSR Journal of Mathematics vol. 11, pp.38-48,2015.

[6] K.Karmina, “Fluid flow and stability analysis in certain thin Liquid films,” M.Sc. thesis, Faculty of Science, University of Zakho, 2013.

[7] K. Mysels and K Shinoda “Soap films, studies of their Thicking and a bibliography”,Pergamon press, London, 1959.

★★★