AN ALGORITHM FOR REDUCING MULTI-CONSTRAINED LINEAR AND QUADRATIC PROGRAMMING PROBLEMS TO SINGLE-CONSTRAINED PROBLEMS

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Abstract- Solving multi-constrained programming problems have always been problematic in the field of mathematical programming. Large constraints always involve a lot of time, caution and space especially when tableaus are involved and until the advent of computer program or software that can provide solutions to the problems in a little space of time but here in this study an algorithm has been developed capable of reducing multi-constrained linear and quadratic programming problems to single constrained programming problems. The work also includes comparison of solutions of the formulated models with the original problems.

Keywords- New Algorithm, Linear, Quadratic, Programming, Multi-Constrained, Single-Contrained

I. INTRODUCTION

Most nonlinear programming problems are developed to solve numerous practical problems in the world of business, Hasan [1]. Nonlinear programming is a mathematical programming technique designed to determine the optimal solutions to many real world problems. In a nonlinear programming problem, the objective function is either non-linear, or a number of the constraints is a nonlinear function or both the constraints and objective function are nonlinear functions, while in linear programming, both the objective function and the constraints are linear functions, Vankova [2].

1.1 Linear Programming Problems

Numerous mathematical programming applications are geared towards or modeled as linear programming problems, Vanderbei [3].

$$\left.\begin{aligned} &\text{Maximize } z = \sum_{j=1}^{n} c_j x_j \\ &\text{subject to:} \\ &\sum_{j=1}^{n} a_{ij} x_j \le b_i, i = 1,2,\dots\dots,m, \\ &\text{and } x_j \ge 0, j = 1,2,\dots\dots n \end{aligned}\right\} \quad 1.1$$

Where $x_j, j = 1,2,\dots\dots n$ are the level of activities or the decision variables;
m is the number of resources;
n is the number of commodity;
c_j is the profit per unit commodity j produced;
a_{ij} is the number of units of resources i required to produce one unit of commodity j;
b_j is the maximum number of units of resources i available.

Nonlinearities in the form of either nonlinear objective functions or nonlinear constraints are significant for representing a real application of a mathematical program. Assumptions made in linear programming may also lead to appropriate problem representations over the range of decision variables being considered.

1.2 Nonlinear Programming Problems

The solutions methods of nonlinear programming can generally be classified as direct or indirect methods. The gradient algorithms can be described as the direct method where the maximum (minimum) of a problem is obtained by following the quickest rate of increase and decrease) of the objective function. Examples of the indirect method where the original problem is replaced by an auxiliary one from which the optimum is determined are the quadratic programming, separable programming, and the stochastic programming, Taha [4].

As implied by Vankova [2], nonlinear programming problems are those in which the objective function and/or its constraints are nonlinear functions of the decision variables. Below are some examples of systems that cannot be described by linear models are as follows:

1. Systems with periodic variations like a digital clock circuit and heart and nerve impulses;
2. Systems with multiple operating points like chemical reactions and buckling beams;
3. Systems sensitive to initial conditions like the dynamics of population, climatic models and fluid flow models referred to as chaotic or complex dynamics.
4. Fixed charge problems, etc.

A general optimization problem is to select n decision variables $x_1, x_2, \dots, x_n$ from a given feasible region in such a way as to optimize (minimize or maximize) a given objective function $f(x_1, x_2, \dots, x_n)$ of the decision variables. The above stated problem is called a nonlinear programming problem (NLP) if the objective function is nonlinear and/or the feasible

region is determined by nonlinear constraints. The general nonlinear program is stated as:

Let z be a real valued function of n variables defined

(1) $z = f(x_1, x_2, \ldots, x_n)$,

Let $(b_1, b_2, \ldots, b_n)$ be a set of constraints such that

(2)
$$\begin{cases} g^1 = (x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}b_1 \\ g^2 = (x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}b_2 \\ \vdots \\ g^m = (x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}b_m \end{cases}$$

Where $g^{i}s$ are real valued functions of n variables $x_1, x_2, \ldots, x_n$

Finally, let

(3) $x_j \geq 0, \quad j = 1, 2, \ldots \ldots m$

Considering the objective function and the constraints, if either or some; or both are non-linear, then the problem of determining the n-type which makes a max/min and satisfies (b) and (c), is called a general nonlinear programming (GNLP) problem. A nonlinear problem may be nonlinearly constrained or linearly constrained. If the objective function of a linearly constrained nonlinear problem is a quadratic function then this problem is said to be a quadratic programming (QP) problem, Hasan [1].

1.3 Quadratic programming problem

The general quadratic programming model can be stated as

$$\max Z = CX - \frac{1}{2}X^{T}QX \quad 1.2$$

$$\text{subject to } AX \leq B \quad 1.3$$

$$X \geq 0 \quad 1.4$$

Where X and B are column vectors, C is a row vector, A and Q are matrices, and the superscript T denotes the transpose of X.

1.4 Aims of the study

The study is designed to create an algorithm for simplifying some all kind of linear or quadratic multi-constraint problems to single constraint kind of same programming problems.

1.5 Objectives of the study

This study shall;

1. Consider linear and quadratic programming problems of not less than one constraints.
2. Provide some numerical solutions to the problems stated.
3. Compare the solutions of the new auxiliary problems (using proposed algorithm) to the solutions provided by a direct method using the Lingo Solver.

1.6 Significance of the study

The study shall provide mathematical programmers an algorithm designed to reduce some multi-constraint linear and quadratic programming problems to single constraint less complex linear or quadratic programming problems which can easily be solved using the Simplex method, Wolfe's method or the Separable programming method.

1.7 Scope

The study is focused at providing an algorithm that reduces multi-constraint linear or quadratic programming problems to single-constraint problems. This therefore, implies, that the study covers only linear and quadratic programming and not all forms of nonlinear programming problems.

2.1 Literature review

Hasan [1], extend the simplex method for solving QP problems by replacing one basic variable at an iteration of the simplex method. And then, developed an algorithm and a computer technique for solving quadratic programming problem involving the product of two indefinite factorized linear functions. The study used MATHEMATICA programming language for developing the technique. We also illustrate numerical examples to demonstrate our technique.

The special type quadratic programming problem was the quasi-concave quadratic programming problem (QP) subject to linear constraints such as:

$$\text{Maximize } Z = (cx + \alpha)(dx + \beta) \quad 2.1$$

$$\text{subject to: } Ax \leq b \quad 2.2$$

$$x \geq 0 \quad 2.3$$

Where $A = (a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_n)$ is a matrix, $b \in \mathcal{R}^m$, and $c, d \in \mathcal{R}^n$, and $\alpha, \beta \in \mathcal{R}$. The study assumed that

(a) $(cx + \alpha)$ and $(dx + \beta)$ are positive for all feasible solution.

(b) The constraints set $x \in S = \{x: Ax = b, x \geq 0\}$ is non-empty and bounded.

Hasan [1], states that a quadratic programming problem is solved starting with some x which satisfies the constraints but does not necessarily maximize Z. Where the variable x is replaced by a series of other $x's$ (which are not necessarily basic feasible solutions) but satisfies the constraints until the required solution is obtained.

Ghadle et al. [5], provided an alternative method for Wolfe's modified simplex method. The study claims that the method is easy to solve quadratic programming problem (QPP) concerned with non-linear programming problem (NLPP). The study went further to provide an algorithm for finding optimal solution of any NLPP by an alternative method for Wolfe's modified simplex method.

Where the quadratic programming problem is stated as:

$$f(x) = \sum_{j=1}^{n} c_j x_j - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{kj} x_k x_j \quad 2.4$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, (\ i = 1, 2, \ldots \ldots, m, \quad j = 1, 2, \ldots, n) \qquad 2.5$$

$$\sum_{j=1}^{n} x_j \geq 0 \qquad 2.6$$

Where $c_{kj} = c_{jk}$, for all $j, k, b_i \geq 0$. and $i = 1, 2, \ldots \ldots, m$.

3.1 Methodology

Here, the algorithm for reducing large multi-constraints linear or quadratic programming problems shall be stated according to the step-by-step procedures as follows:

The study shall modify the problem in equation (1.3) such that;

$$\max f_1(x) = CX - \frac{1}{2} X^T Q X \qquad 3.1$$

$$\text{subject to: } g_m(x) = \sum_{j=1}^{n} a_{ij} x_j \leq b_i, (\ i = 1, 2, . m, j = 1, 2, , n) \qquad 3.2$$

$$x_j \geq 0, \quad j = 1, 2, \ldots \ldots, n \qquad 3.3$$

Let the number of variables be $j = 1, 2, 3, \ldots \ldots, n$, such that $x_j = x_1, x_2, x_3, \ldots \ldots x_n$ and number of constraints $i = 1, 2, 3, \ldots \ldots, m$, such that $g_i = g_1, g_2, g_3, \ldots \ldots g_m$) represent the constraints.

Step 1. If not all constraint contains all the original variables, then ear-mark the constraint containing all the variables and avoid reducing. (example g_m contains all variable, then, we take note of g_m).

Step 2. Add slack variables to equate the inequality constraints (inequality g_i)

Step 3. In g_1, make x_1 subject formula and substitute for the resultant x_1 in the remaining g_is. This will automatically terminate the g_1 constraint from the problem leaving us with new (g_2, $g_3, \ldots \ldots g_m$) constraints.

Step 4. In the new g_2, make x_2 subject formula and substitute for the resultant x_1 in the remaining g_is. This will automatically terminate the g_1 constraint from the problem leaving us with yet another new ($g_3, \ldots \ldots g_m$) constraints.

Step 5. Continue with this procedure until the ear-marked g_i (example g_m) is the only new constraint remaining in the problem.

Step 6. Then, substitute for all the x_js in the objective function. This will produce a new objective function.

Step 7. Formulate the new quadratic problem using the new objective function in step 6 and the new constraint in step 5.

3.2 Observations

1. The final problem (resultant problem) can contain just one of the original variables or not. This implies that at the end, most of the original variables (x_js) might not be contained in the final model which is mostly occupied by the slack variables.
2. Solutions to the problem might be represented by the slack variables but values of the original variables can be calculated by substituting for the slacks in the various derived equation representing the slacks during the formulation of the problem.
3. The procedure reduces time spent dealing with the numerous constraint of quadratic problems especially when the separable programming technique or the Wolfe's method are to be used.
4. The new objective function and constraint might look a little complex but are not especially if done by someone with the knowledge of using the Wolfe's method.
5. Not necessarily all slack variables in the last constraint will be in the objective function.
6. This technique is easy to follow and capable of reducing large constrained linear or quadratic programming problems to single constrained problems.

3.3 Results and Discussion

Here, the study shall show examples of how the algorithm is applied to problems.

Problem 1

The Wyndor Glass problem, Hillier et al. [6].

$$\max f(x) = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$$

$$\text{subject to: } x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

A look at the problem, it is seen that the problem contains three (3) constraints, therefore, we have g_1, g_2, and g_3.

Thus, Let

g_1: $x_1 \leq 4$ (i)

g_2: $2x_2 \leq 12$ (ii)

g_3: $3x_1 + 2x_2 \leq 18$ (iii)

Step 1. If not all constraint contains all the original variable, then ear-mark the constraint containing all the variables and avoid reducing it.

Only the third constraint contains all the original variables, so we take note of

$3x_1 + 2x_2 \leq 18$

Step 2. Add slack variables to equate the inequality constraints (inequality g_i)

There are three (3) constraints so the slacks variables shall be three (3).

g_1: $x_1 + s_1 = 4$ 1a(i)

g_2: $2x_2 + s_2 = 12$ 1a(ii)

g_3: $3x_1 + 2x_2 + s_3 = 18$ 1a(iii)

Step 3. In g_1, make x_1 subject formula and substitute for the resultant x_1 in the remaining g_is.

Thus, $x_1 = 4 - s_1$ 1b(i)

Equation 1a(ii) does not contain x_1, hence, we move to 1a(iii)

g_3: $3(4 - s_1) + 2x_2 + s_3 = 18$ 1b(ii)

Step 4. In the new g_2, make x_2 subject formula and substitute for the resultant x_2 in the remaining g_is.

Thus,

$x_2 = \frac{(12 - s_2)}{2}$ 1c(i)

Equation 1b(ii) becomes

$$g_3: \ 3(4 - s_1) + 2\left(\frac{12 - s_2}{2}\right) + s_3 = 18 \quad 1c(ii)$$

g_3: $-3s_1 - s_1 + s_3 = -6$ 1c(iii)

Step 5. Continue with this procedure until the ear-marked g_i (example g_m) is the only new constraint remaining in the problem.

Since g_3: $-3s_1 - s_1 + s_3 = -6$ is the only remaining constraint in the problem, we proceed to step 6.

Step 6. Substitute for all the x_js in the objective function. This will produce a new objective function.

$$h(x) = 126(4 - s_1) - 9(4 - s_1)^2 + 182\left(\frac{12 - s_2}{2}\right) - 13\left(\frac{12 - s_2}{2}\right)^2$$

One can decide to expand the above equation further but in this study, we shall not.

Step 7. Formulate the new quadratic problem using the new objective function in step 6 and the new constraint in step 5.

New Wyndor Glass problem

$$\max h(x) = 126(4 - s_1) - 9(4 - s_1)^2 + 182\left(\frac{12 - s_2}{2}\right) - 13\left(\frac{12 - s_2}{2}\right)^2$$

subject to:

$$-3s_1 - s_1 + s_3 = -6$$
$$s_1, s_2, s_3 \geq 0$$

Comparison of Solutions

Solution to the Wyndor Glass problem using the Lingo Solver is represented as;

Local optimal solution found.
Objective value: 857.0000
Infeasibilities: 0.000000
Extended solver steps: 5
Total solver iterations: 31

Model Class: NLP

Total variables: 2
Nonlinear variables: 2
Integer variables: 0

Total constraints: 6
Nonlinear constraints: 1

Total nonzeros: 8
Nonlinear nonzeros: 2

Variable	Value	Reduced Cost
X1	2.666667	0.000000
X2	5.000000	0.000000

Row	Slack or Surplus	Dual Price
1	857.0000	1.000000
2	1.333333	0.000000
3	2.000000	0.000000
4	0.000000	26.00000
5	2.666667	0.000000
6	5.000000	0.000000

Here, we see that $x_1 = 2.666667$ and $x_2 = 5.000000$.

Let's see what the solution to the new Wyndor problem is using the Lingo Solver

Local optimal solution found.
Objective value: 857.0000
Infeasibilities: 0.000000
Extended solver steps: 5
Total solver iterations: 33

Model Class: NLP

Total variables: 3
Nonlinear variables: 2
Integer variables: 0

Total constraints: 5
Nonlinear constraints: 1

Total nonzeros: 8
Nonlinear nonzeros: 2

Variable	Value	Reduced Cost
S1	1.333333	0.000000
S2	2.000000	0.000000
S3	0.000000	0.000000

Row	Slack or Surplus	Dual Price
1	857.0000	1.000000
2	0.000000	26.00000
3	1.333333	0.000000
4	2.000000	0.000000
5	0.000000	-26.00000

Here, we see that $s_1 = 1.333333$, $s_2 = 2.0000000$ and $s_3 = 0$. Therefore, we substitute for s_1 and s_2 in 1b(i) and 1c(i).

Thus, $x_1 = 4 - 1.333333 = 2.666667$ and $x_2 = \frac{(12-2)}{2} = 5.000000$.

Hence, we conclude that the Old Wyndor Glass problem and the New problem produced the same result.

Problem 2

$$\min f(x) = 3x_1 + 4x_2$$
$$\text{subject to: } x_1 + 2x_2 \geq 10$$
$$4x_1 + 3x_2 \geq 24$$
$$x_1 + 5x_2 \geq 18$$
$$x_1, x_2 \geq 0$$

The problem is that of three (3) constraints, therefore, hence

We have

$g_1: -x_1 - 2x_2 \leq -10$ (i)

$g_2: -4x_1 - 3x_2 \leq -24$ (ii)

$g_3: -x_1 - 5x_2 \leq -18$ (iii)

Step 1. All constraint contains the original variable, so we ear-mark g_2 for non-reduction.

$$g_2: -x_1 - 5x_2 \leq -18$$

Step 2. Add slack variables to the constraints.

The new constraints becomes;

$g_1: -x_1 - 2x_2 + s_1 = -10$ 2a(i)

$g_2: -4x_1 - 3x_2 + s_2 = -24$ 2a(ii)

$g_3: -x_1 - 5x_2 + s_3 = -18$ 2a(iii)

Step 3. In 3a(i), make x_1 subject formula and substitute for the resultant x_1 in the remaining g_2 and g_3.

Here we have, $x_1 = 10 - 2x_2 + s_1$ 2b(i)

Hence, 3a(ii) and 3a(iii) becomes

$g_2: 4(10 - 2x_2 + s_1) - 3x_2 + s_2 = -24$

$g_2: 5x_2 - 4s_1 + s_2 = 16$ 2b(ii)

And $g_3: -(10 - 2x_2 + s_1) - 5x_2 + s_3 = -18$

$g_3: -3x_2 - s_1 + s_3 = -8$ 2b(iii)

Step 4. In 2b(ii), make x_2 subject formula and substitute the resultant x_2 into 2b(i) and 2b(iii).

Thus,

$$x_2 = \left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right),$$

$$x_1 = 10 - 2\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) + s_1 \quad 2c(i)$$

$$x_2 = \left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right)$$

Therefore, equation 2b(iii) becomes

$$g_3: -3\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) - s_1 + s_3 = -8, \quad 2c(ii)$$

Step 5. Since

$$g_3: -3\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) - s_1 + s_3 = -8$$

is the last remaining constraint in the problem, hence, we proceed to step 6.

Step 6. Substitute for x_1 and x_2 in the objective function.

$$h(x) = 3\left(10 - 2\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) + s_1\right) + 4\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right)$$

Step 7. We can now proceed to formulate the new quadratic programming problem using the new objective function in step 6 and the new constraint in step 5.

$$\max h(x) = 3\left(10 - 2\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) + s_1\right) + 4\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right)$$
$$\text{subject to: } -3\left(\frac{16}{5} + \frac{4s_1}{5} - \frac{s_2}{5}\right) - s_1 + s_3 = -8$$
$$s_1, s_2, s_3 \geq 0$$

The solution to the original problem using the Lingo Solver is;

Global optimal solution found.

Objective value:	23.60000
Infeasibilities:	0.000000
Total solver iterations:	2
Model Class:	LP
Total variables:	2
Nonlinear variables:	0
Integer variables:	0
Total constraints:	6
Nonlinear constraints:	0
Total nonzeros:	10
Nonlinear nonzeros:	0

Variable	Value	Reduced Cost
X1	3.600000	0.000000
X2	3.200000	0.000000

Row	Slack or Surplus	Dual Price
1	23.60000	-1.000000
2	0.000000	-1.400000
3	0.000000	-0.4000000
4	1.600000	0.000000
5	3.600000	0.000000
6	3.200000	0.000000

Where $x_1 = 3.6$, $x_2 = 3.2$ and $h(x) = 23.600$

The solution to the new problem using the Lingo Solver is;

Global optimal solution found.

Objective value:	23.60000
Infeasibilities:	0.000000
Total solver iterations:	0
Model Class:	LP
Total variables:	3
Nonlinear variables:	0
Integer variables:	0

Total constraints:	5
Nonlinear constraints:	0
Total nonzeros:	8
Nonlinear nonzeros:	0

Variable	Value	Reduced Cost
S1	0.000000	1.400000
S2	0.000000	0.4000000
S3	1.600000	0.000000

Row	Slack or Surplus	Dual Price
1	23.60000	-1.000000
2	0.000000	0.000000
3	0.000000	0.000000
4	0.000000	0.000000
5	1.600000	0.000000

Where $s_1 = 0.0$, $s_2 = 0.0$, $s_2 = 1.60$ and $h(x) = 23.600$

DISCUSSION

The algorithm proves to be of relevance in problems involving large multi-constrained linear and quadratic programming problems just as illustrated in the solved examples. The issue of convexity does not matter in using the technique as both convex and non-convex functions can still be subjected to the technique without violation of the original problem.

CONCLUSION

This algorithm is unique and can go a long way in reducing the burden of having got to consider numerous constraints in using the simplex method in solving a linear programming problem with numerous constraints or formulating a separable programming problem or even using the Wolfe's method in solving a multi-constrained quadratic programming problem.

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