COMMON FIXED POINT THEOREMS IN COMPLEX VALUED B-METRIC SPACES

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Abstract - Azam et. al.(2011) introduced the notion of complex valued metric spaces and obtained common fixed point result for mappings in the context of complex valued metric space. Rao et. al. (2013) introduced the notion of complex valued b-metric space. Badshah et. al (2017) established some fixed point results for a pair of compatible mappings satisfying a rational inequality in complex valued metric spaces. In this paper, we study complex valued b-metric space and generalize the result given by Azam et. al[1] and Badshah et. al [2]

Index Terms - Common fixed point, contractive type mapping, complex valued b-metric spaces.
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I. INTRODUCTION

One of the most influential spaces is complex valued b-metric spaces, introduced by Rao et.al. [11] in 2013, which was more general than the complex valued metric spaces [1]. They proved some fixed point results for rational type mappings in complex valued b-metric spaces. Since then, this notion has been used by many authors to obtain various fixed point theorems (see [2],[3],[4],[5],[6],[7],[8],[9], [11]).
The aim of this paper is to prove common fixed point theorem for two self-mappings in a complete complex valued b-metric space.

II. PRELIMINARIES

Let us start by defining some important notations and definitions.
Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(1) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;
(2) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;
(3) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$;
(4) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.
In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied, also we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that
(a) if $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$;
(b) if $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$;
(c) if $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in C_+$.
The following definition is recently introduced by Rao et. al.[11].
Definition 2.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.
The pair (X,d) is called a complex valued b-metric space.
Example 2.2. If $X = [0,1]$, define the mapping $d: X \times X \to \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ for all $x, y \in X$. Then (X,d) is a complex valued b-metric space with $s = 2$.
Definition 2.3. Let (X,d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X: d(x,y) \prec r\} \subseteq A$.
(ii) A point $x \in X$ is called limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \emptyset$.
(iii) A subset $A \subseteq X$ is called open set whenever each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is called closed set whenever each element of A belongs to A.
(v) The family $F = \{B(x,r): x \in X \text{ and } 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.
Definition 2.4. Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$.
(i) If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that for all $n > N, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent andconverges to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.
(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that for all $n > N, d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(iii) If every Cauchy sequence in X is convergent in X, then (X,d) is said to be a complete complex valued b-metric space.
Lemma 2.5 . Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.6 . Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Here, we give some notions in fixed point theory.

Definition 2.7: Let S and T be self-mappings of a nonempty set X.

(i) A point $x \in X$ is said to be a fixed point of T if Tx=x.

(ii) A point $x \in X$ is said to be a coincidence point of S and T if Sx=Txand we call $w = Sx = Tx$, a point of coincidence of S and T.

(iii) A point $x \in X$ is said to be a common fixed point of S and T if Sx=Tx.

In 1976, Jungck [12] introduced concept of commuting mappings as follows:

Definition 2.8: Let X be a non-empty set. The mappings S and T are commuting if $TSx = STx$ for all $x \in X$.

Afterward, Sessa [18] introduced concept of weakly commuting mappings which are more general than commuting mappings as follows:

Definition 2.9: Let S and T be mappings from a metric space (X, d) into itself. The mappings S & T are said to be weakly commuting of $d(STx, TSx) \leq d(Sx, Tx)$ for all $x \in X$.

In 1986 Jungck[13] introduced the more generalized commuting mapping in metric space called compatible mappings, which also are more general than the concept of weakly commuting mappings as follows.

Definition 2.10: Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence is X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.

Remark 1.1: In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true.

III. MAIN RESULT

Theorem 3.1. Let (X, d) be a complete complex valued b-metric space and the mappings $S, T: X \to X$ are self mappings satisfying the condition

$$d(Sx, Ty) \leq \lambda \frac{d(x, Sx) + d(x, Ty) + d(y, Ty)d(y, Sx)}{d(x, Ty) + d(y, Sx)} + \mu d(x, y) \quad (2.1)$$

for all $x, y \in X$, where $d(x, Ty) + d(y, Sx) \neq 0 \lambda, \mu$ are nonnegative reals with the condition $\lambda + \mu < 1$ also if one of S or T is continuous and the pair (S, T) is compatible, then S and T have a unique common fixed point.

Proof: Let $x_o \in X$ be an arbitrary point and we define

$$x_{2n+1} = Sx_{2n},$$
$$x_{2n+2} = Tx_{2n+1}, \text{for } n = 0,1,2,3 \ldots \ldots (2.2)$$

By equations (2.1) and (2.2)

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n+1}, Tx_{2n+1})$$
$$\leq \lambda \frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})} + \mu d(x_{2n}, x_{2n+1})$$
$$\leq \lambda \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} + \mu d(x_{2n}, x_{2n+1})$$

so

$$d(x_{2n+1}, x_{2n+2}) \leq \lambda \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+2})} + \mu d(x_{2n}, x_{2n+1})$$
$$\leq (\lambda + \mu) d(x_{2n}, x_{2n+1})$$

similarly

$$d(x_{2n+2}, x_{2n+3}) = d(Sx_{2n+1}, Tx_{2n+2})$$
$$\leq \lambda \frac{d(x_{2n+1}, Sx_{2n+1})d(x_{2n+1}, Tx_{2n+2}) + d(x_{2n+2}, Tx_{2n+2})d(x_{2n+2}, Sx_{2n+1})}{d(x_{2n+1}, Tx_{2n+2}) + d(x_{2n+2}, Sx_{2n+1})} + \mu d(x_{2n+1}, x_{2n+2})$$
$$\leq \lambda \frac{d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, x_{2n+3}) + d(x_{2n+2}, x_{2n+3})d(x_{2n+2}, x_{2n+2})}{d(x_{2n+1}, x_{2n+3}) + d(x_{2n+2}, x_{2n+2})} + \mu d(x_{2n+1}, x_{2n+2})$$
$$\leq (\lambda + \mu) d(x_{2n+1}, x_{2n+2})$$

Now let $h = (\lambda + u)$

Then

$$d(x_{n+1}, x_{n+2}) \leq h d(x_n, x_{n+1}) \leq h^2 d(x_{n-1}, x_n) \leq h^3 d(x_{n-2}, x_{n-1})$$

so

$$d(x_{n+1}, x_{n+2}) \leq h^{n+1} d(x_0, x_1)$$

So for any $m > n$ we have

$$d(x_n, x_m) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)]$$
$$= s d(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)]$$
$$\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + - - + s^{m-n} d(x_{m-1}, x_m)$$
$$\leq s h^n d(x_0, x_1) + s^2 h^{n+1} d(x_0, x_1) + s^3 h^{n+2} d(x_0, x_1) + - -$$
$$- + s^{m-n} h^{m-1} d(x_0, x_1)$$
$$\leq \frac{s^{n-1}}{s^{n-1}} [s h^n d(x_0, x_1) + s^2 h^{n+1} d(x_0, x_1) + ___ + s^{m-n} h^{m-1} d(x_0, x_1)]$$
$$\leq \frac{1}{s^{n-1}} [s^n h^n + s^{n+1} h^{n+1} + - - + s^{m-1} h^{m-1}] d(x_0, x_1)$$
$$= \frac{1}{s^{n-1}} \frac{s^n h^n}{1 - Sh} d(x_0, x_1)$$
$$= \frac{s h^n}{1 - s^h} (d(x_0, x_1))$$

So $|d(x_m, x_n)| \leq \left[\frac{s h^n}{1 - sh}\right] d(x_0, x_1) \to 0$ as $m, n \to \infty$.

It implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete $\exists u \in X$ such that $x_n \to u$. Consequently the sequences Sx_{2n} and Tx_{2n+1} also converges to u.

Let S is continuous. Since S & T are compatible maps then $\lim_{n\to\infty} d(STx_{2n}, TSx_{2n}) = 0$.

Also since S is continuous $STx_{2n} \to Su$

Now (STx_{2n}, Tx_{2n+1})

$$\leq \lambda\left[\frac{d(Tx_{2n},STx_{2n})d(Tx_{2n},Tx_{2n+1})+d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},STx_{2n})}{d(Tx_{2n}Tx_{2n+1})+d(x_{2n+1}STx_{2n})}\right]$$
$$+\mu d(Tx_{2n},x_{2n+1})$$

Taking limit $n\to\infty$ both sides we have

$$d(Su,u)\leq\lambda\frac{d(u,Su)d(u,u)+d(u,u)d(u,Su)}{d(u,u)+d(u,su)}$$
$$+\mu d(u,u)$$

It shows that $Su=u$

Suppose if not $d(u,Su)>0=z$ say then we have

$$z\leq s[d(u,x_{2n+2})+d(x_{2n+2},Su)]$$
$$\leq s[d(u,x_{2n+2})+d(Su,Tx_{2n+1})]$$
$$\leq sd(u,x_{n+2})+sd(Su,Tx_{2n+1})]$$
$$\leq sd(u,x_{2n+2})$$
$$+s\{\lambda\frac{[d(u,Su)d(uTx_{2n+1})+d(x_{2n+1})d(x_{2n+1}Su)]}{d(u,Tx_{2n+1})+d(x_{2n+1}Su)}$$
$$+\mu d(u,x_{2n+1})\}$$

Taking modulus of both sides and limit as $n\to\infty$ we get

$$|Z|\leq|sd(u,x_{2n+2})|+$$
$$+\frac{s\lambda\{|d(u,Su)\;||d(u,x_{2n+2})\;|+|d(x_{2n+1},x_{2n+2})||d(x_{2n+1}Su)|}{|d(u,x_{2n+2})|+|d(x_{2n+1}Su)|}$$

Taking limit $n\to\infty$ we get

$|Z|\leq 0$ - it is a contradiction as $d(u,Su)=z>0$

Hence $u=Su$.

Similarly$u=Tu$.

We will show that S and T have a unique common fixed point.

Suppose u^* is another fixed point of S and T Then by (2.1) we have

$$d(u,u^*)=d(Su,Tu^*)$$
$$\leq\lambda\frac{d(u,Su)d(u,Tu^*)+d(u^*,Su)d(u^*,Tu^*)}{d(u,Tu^*)+d(u^*,Su)}$$
$$+ud(u,u^*)$$

So $d(u,u^*)\leq\mu d(u,u^*)$

Now $\mu\geq1$ so it is a contradiction.

This shows that $u=u^*$

So S and T have a unique common fixed point. This proves the theorem.

Theorem 3.2 – Let (X,d) be a complete complex valued b-metric and let S and T are self mappings $S,T:X\to X$ satisfying the condition.

$$d(Sx,Ty)\leq\lambda\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}$$
$$+\mu d(x,y)$$

for all $x,y\in X$, where $d(x,Ty)+d(y,Sx)\neq0$ and λ and μ are nonnegative reals with $\lambda+\mu<1$. Then S *and* T have a unique common fixed point.

Proof. This can be proved by Theorem 3.1.

Corollary 3.1 Let (X,d) be a complete complex valued b-metric space and let $T:X\to X$ satisfy the condition

$$d(Tx,Ty)\leq\lambda\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}+\mu d(x,y)(2.3)$$

for all $x,y\in X$, where $d(x,Ty)+y(y,Tx)\neq0$ and λ and μ are nonnegative real numbers with $\lambda+\mu<1$. Then T has a unique fixed point.

Proof. This can be proved by putting $S=T$in the theorem 3.2.

Corollary 3.2. Let (X,d) be a complete complex valued b-metric space and $T:X\to X$ satisfy the condition.

$$d(T^nx,T^ny)$$
$$\leq\frac{d(x,T^nx)d(x,T^ny)+d(y,T^ny)d(y,T^nx)}{d(x,T^ny)+d(y,T^nx)}+\mu d(x,y)$$

for all $x,y\in X$, where $d(x,T^ny)+d(y,T^nx)\neq0$ and λ,μ are nonnegative real number with $\lambda+\mu<1$ then T has a unique fixed point.

Proof. This can be proved by Theorem 3.1, 3.2 and Corr. 3.1.

The above theorems and corollaries are the generalization and extension of common fixed point theorems in the recent literature of complex value metric spaces

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