NUMERICAL SOLUTION OF MLPG METHOD BASED ON REPRODUCTING KERNEL PARTICLE INTERPOLATION (RKPI) FOR SOLVING CONVECTION-DIFFUSION EQUATION

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Abstract - The convection-diffusion equation with the non-local boundary conditions arises in many physical phenomena. In this paper, the meshless local Pretrov-Galerkin (MLPG) method based on the reproducing kernel particle interpolation (RKPI) method to solving time-dependent convection-diffusion equation in two-dimensional spaces on a square domain. A technique is proposed to construct shape functions using reproducing kernel function, and the Heaviside step function is used as a test function in each sub-domain to avoid the need for domain integral in local symmetric weak form. The time derivatives are approximated by the Crank-Nicolson method. Numerical experiment is implemented to verify the efficiency, easiness and accuracy of the method. The method can be used to achieve relatively high accuracy, and the essential boundary conditions can be directly enforced.

Keywords - MLPG; RKPI, Convection-Diffusion Equation, Crank-Nicolson, Non-Local Boundary Condition.

I. INTRODUCTION

Considerable research works have been carried out on partial differential equations with non-local boundary conditions due to their importance in many fields of science and engineering applications such as chemical diffusion, heat conduction processes population dynamics, thermo-elasticity, medical science, electrochemistry and control theory. Because of the complex problems, some analytic solutions are difficult to find in the sense of boundary conditions and geometrical shapes.

There are three methods that are wellknown as finite difference methods (FDM), finite volume methods (FVM)and finite element methods (FEM) which are similar process on first steprequired to create a mesh or element on the domain. If the domain has asimple shape such as circle, rectangle or cubes, then mesh is not difficult to create. Thenumerical method has been one of the most popular methods recently is meshless method. It worked very well for the convection-diffusion equations subject to non-local boundary condition. The main objective of the meshless methods is to reduce the problem to create the mesh. Meshless local Petrov-Galerkin (MLPG) method is truly meshless method. The MLPG method is based on a weak form and the reproducing kernel particle method(RKPM).The RKPM of interpolation is developed by [8].The method of performing an integral transformation with a corrected kernel function is called the Reproducing Kernel Particle Method (RKPM), since it satisfies the reproduction conditions and therefore exactly reproduces polynomials.

The purpose of this article is to present very efficient meshless method based onRKPM for solving the two-dimensional time dependent convection –diffusion equation with non-local boundary conditions [15].A technique is proposed to construct shape functions using reproducing kernel function, and the Heaviside step function is used as a test function in each sub-domain to avoid the need for domain integral in local symmetric weak form. Problem is set a time-stepping scheme to overcome the time derivative using the Crank-Nicolson method of approximation. Test problem is presented to compare and verify the efficiency and accuracy of the methods in this section. Also, the relative error is obtained to show the convergence of the method.The governing equation for solving the following two-dimensional convection-diffusion equation subject to non-local boundary conditions inΩ can be written as:

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\frac{\partial T}{\partial x_j}=\frac{\partial}{\partial x_j}\cdot\left(\mathbf{K}\frac{\partial T}{\partial x_j}\right)+f,\quad \mathbf{x}\in\Omega, t>0 \quad (1)$$

where T is the unknown function, $\mathbf{K}$ is the diffusivity coefficient, $\mathbf{u}$ is the flow velocity, f is a source term. With initial condition

$$T(x,y,0)=f(x,y),\quad 0\le x,y\le 1. \quad (2)$$

The boundary conditions are assumed to be:
-The non-local boundary condition:

$$T(x,0,t)=h_0(x)\mu(t),\quad 0\le t\le \mathbf{T_{max}}\quad \mathbf{x}\in\Gamma_n. \quad (3)$$

-The essential boundary condition:

$$T=h_1(\mathbf{x},t),\quad 0\le t\le \mathbf{T_{max}}\qquad \mathbf{x}\in\Gamma_d. \quad (4)$$

-The natural boundary condition:

$$\mathbf{n}\cdot\frac{\partial T}{\partial x_j}=g(\mathbf{x},t),\quad 0\le t\le \mathbf{T_{max}}\qquad \mathbf{x}\in\Gamma_q. \quad (5)$$

And the integral condition

$$\int_\Omega T(\mathbf{x},t)\,d\Omega = m(t),\ 0\le x\le 1,\quad 0\le y\le 1, \qquad (6)$$

where $\mu(t)$ is the unknown function, g, h_0, h_1 and m are given functions. The non-local boundary condition is variable-separable, with spatial dependence given by $h_0(\mathbf{x})$ and time dependence given by $\mu(t)$. $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, where $\partial\Omega$ is the boundary of the domain. Γ_n, Γ_d and Γ_q are the part of the boundary $\partial\Omega$, satisfying $\Gamma_n\cap\Gamma_d\cap\Gamma_q=\phi$ and $\Gamma_n\cup\Gamma_d\cup\Gamma_q=\partial\Omega$.

II. NUMERICAL METHODOLOGY

2.1. Reproducing kernel particle interpolation

The RKPM of interpolation is developed by [8]. The method of performing an integral transformation with a corrected kernel function is called the Reproducing Kernel Particle Method (RKPM), since it satisfies the reproduction conditions and therefore exactly reproduces polynomials. The reproducing conditions are those requirements that must be fulfilled, in order for the basic functions to be reproduced exactly.

The reproducing kernel particle interpolation $u^h(\mathbf{x})$ is defined by:

$$u^h(\mathbf{x})=\sum_{I=1}^{N}C(\mathbf{x},\mathbf{x}_I)\,w(\mathbf{x}-\mathbf{x}_I)\,u(\mathbf{x}_I)\,c=\sum_{I=1}^{N}\chi^I(\mathbf{x})u(\mathbf{x}_I) \quad (7)$$

Where the shape function in the RKPM interpolation can be written as

$$\chi^I(\mathbf{x})\equiv\chi^I(\mathbf{x}-\mathbf{x}_I)=C(\mathbf{x},\mathbf{x}_I)\,w(\mathbf{x}-\mathbf{x}_I)\Delta\Omega_I \quad (8)$$
$$=D(\mathbf{x}-\mathbf{x}_I)\,w(\mathbf{x}-\mathbf{x}_I)$$

Here, there is no summation on I. The function $D(\mathbf{x}-\mathbf{x}_I)$ in the shape function can be derived from the so-called "reproducing conditions".

$$D(\mathbf{x}-\mathbf{x}_I)=\sum_{j=1}^{3}d_j(\mathbf{x})q_j(\mathbf{x}_I)=\mathbf{d}^T(\mathbf{x})\mathbf{q}(\mathbf{x}_I) \quad (9)$$

where $q_j(\mathbf{x})$ are three linearly independent functions. Then, the shape functions can be written as:

$$\chi^I(\mathbf{x})=\mathbf{d}^T(\mathbf{x})\mathbf{q}(\mathbf{x}_I)\,w(\mathbf{x}-\mathbf{x}_I) \qquad (10)$$

To ensure the positive definiteness of the equations for d_j, it is desirable to let $\mathbf{q}(\mathbf{x}_I)=\mathbf{p}(\mathbf{x}_I)$. Combining equation (9) with a set of three linear algebraic equations, we obtain

$$\mathbf{Ad}=[1,0,0]^T \quad (11)$$

where $\mathbf{A}$ is the moment matrix

$$A_{ij}=\sum_{k=1}^{N}w_k(\mathbf{x})\,p_i(\mathbf{x}_k)\,p_j(\mathbf{x}_k). \qquad (12)$$

To prove that $\chi^I(\mathbf{x})$ computed by equation (4) is equal to $\Phi^I(\mathbf{x})$ calculated from equation $\Phi^I(\mathbf{x})=\mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})$ for the case $\mathbf{q}(\mathbf{x})=\mathbf{p}(\mathbf{x})$, we note that equation (10) can be rewritten as

$$\chi^T(\mathbf{x})=\mathbf{d}^T(\mathbf{x})\mathbf{B} \quad (13)$$

where $\chi^T(\mathbf{x})=\left[\chi^1(\mathbf{x}),\chi^2(\mathbf{x}),...,\chi^N(\mathbf{x})\right]$ (14a)

and $\mathbf{B}(\mathbf{x})=\mathbf{P}^T\mathbf{W}=\left[w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1),...,w_N(\mathbf{x})\mathbf{p}(\mathbf{x}_N)\right]$ (14b)

The consistency conditions can be written as;

$\chi^T(\mathbf{x})\,\mathbf{P}=\mathbf{p}^T(\mathbf{x})$ (15)

and

$\mathbf{P}=\left[\mathbf{p}^T(\mathbf{x}_1)\ \ \mathbf{p}^T(\mathbf{x}_2)\ \ \cdots\ \ \mathbf{p}^T(\mathbf{x}_N)\right]^{\mathbf{T}}=(N\times m)\,matrix.$

Then from equations (14b) and (15)

$$\mathbf{d}^T\mathbf{BP}=\mathbf{d}^T\mathbf{P}^T\mathbf{WP}=\mathbf{d}^T\mathbf{A}=\mathbf{P}^T \qquad (16)$$

where $\mathbf{A}$ and $\mathbf{W}$ as defined by

$$\mathbf{A}(\mathbf{x})=\mathbf{P}^T\mathbf{WP}=\mathbf{B}(\mathbf{x})\mathbf{P}=\sum_{I=1}^{N}w_I(\mathbf{x})\mathbf{p}(\mathbf{x}_I)\mathbf{p}^T(\mathbf{x}_I) \quad (17a)$$

and $\mathbf{W}=\begin{bmatrix} w_1(\mathbf{x}) & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & w_N(\mathbf{x}) \end{bmatrix}=(N\times N)\,matrix,$

respectively. (17b)

This proves that $\mathbf{A}$ in equation (11) is the same as that obtained by equation (17a). Now, if we substitute α back into equation (15), we obtain

$$\chi^T(\mathbf{x})=\mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}=\Phi^T(\mathbf{x}) \quad (18)$$

III. FORMULATION FOR CONVECTION-DIFFUSION EQUATION.

3.1 Local weak formulation

In the present, the MLPG method constructs the weak form over local sub-domains such as Ω_s, which is located entirely inside the global domain Ω. In this

paper they are taken to be of circular shape. Because the weak form is constructed over local sub-domains, the formulation is called the ''local weak formulation''. The local weak form of equation(1) for $\mathbf{x}_i = \left(x^i, y^i\right) \in \Omega_s^i$ can be written as follows:

$$\int_{\Omega_s^i} \left(\frac{\partial T}{\partial t} + \mathbf{u}.\frac{\partial T}{\partial x_j} - \frac{\partial}{\partial x_j}.\left(\mathbf{K}\frac{\partial T}{\partial x_j} \right) \right) v_i d\mathbf{\Omega} = \int_{\Omega_s^i} f v_i d\mathbf{\Omega} \quad (19)$$

where T is the trial function and v_i is a test function. Using the divergence theorem and

$$\frac{\partial}{\partial x_j}.\left(\mathbf{K}\frac{\partial T}{\partial x_j} \right) v_i = \frac{\partial}{\partial x_j}.\left(\mathbf{K}\frac{\partial (T) v_i}{\partial x_j} \right) - \mathbf{K}\frac{\partial T}{\partial x_j}.\frac{\partial v_i}{\partial x_j}$$

yields the following expression:

$$\begin{aligned} &\int_{\Omega_s^i} \frac{\partial T}{\partial t} v_i d\mathbf{\Omega} + \int_{\Omega_s^i} \mathbf{u} \cdot \frac{\partial T}{\partial x_j} v_i d\mathbf{\Omega} \\ &- \int_{\partial\Omega_s^i} \mathbf{K} v_i \frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} + \int_{\Omega_s^i} \mathbf{K}\frac{\partial T}{\partial x_j}.\frac{\partial v_i}{\partial x_j} d\mathbf{\Omega} = \int_{\Omega_s^i} f v_i d\mathbf{\Omega}. \end{aligned} \quad (20)$$

In general, the boundary $\partial\Omega_s^i$ of the local sub-domain Ω_s^i may intersect with the boundary of the global domain Ω. Therefore, $\partial\Omega_s^i = \mathbf{\Gamma}_{sn}^i \cup \mathbf{\Gamma}_{sd}^i \cup \mathbf{\Gamma}_{sq}^i \cup \mathbf{L}_s^i$ where, $\mathbf{L}_s^i$ is a part of $\partial\Omega_s^i$. By imposing the natural boundary condition, we obtain

$$\begin{aligned} &\int_{\Omega_s^i} \frac{\partial T}{\partial t} v_i d\mathbf{\Omega} + \int_{\Omega_s^i} \mathbf{u} \cdot \frac{\partial T}{\partial x_j} v_i d\mathbf{\Omega} - \int_{\Gamma_{sn}^i} \mathbf{K} v_i \frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} \\ &- \int_{\Gamma_{sd}^i} \mathbf{K} v_i \frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} - \int_{\mathrm{L}_s^i} \mathbf{K} v_i \frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} + \int_{\Omega_s^i} \mathbf{K}\frac{\partial T}{\partial x_j}.\frac{\partial v_i}{\partial x_j} d\mathbf{\Omega} \\ &= \int_{\Omega_s^i} f v_i d\mathbf{\Omega} + \int_{\Gamma_{sq}^i} g v_i d\mathbf{\Gamma}. \end{aligned} \quad (21)$$

The test function in each sub-domain is chosen as the Heaviside step function.

$$v_i(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_s^i \\ 0, & \mathbf{x} \notin \Omega_s^i \end{cases} \quad (22)$$

so, $\frac{\partial v}{\partial x_j}(\mathbf{x}) = 0, \forall \mathbf{x} \in \Omega_s^i$ and the local weak form equation (23) is transformed into the following simple local integral equation:

$$\begin{aligned} &\int_{\Omega_s^i} \frac{\partial T}{\partial t} d\mathbf{\Omega} + \int_{\Omega_s^i} \mathbf{u} \cdot \frac{\partial T}{\partial x_j} d\mathbf{\Omega} - \int_{\Gamma_{sn}^i} \mathbf{K}\frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} \\ &- \int_{\Gamma_{sd}^i} \mathbf{K}\frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} - \int_{\mathrm{L}_s^i} \mathbf{K}\frac{\partial T}{\partial \mathbf{n}} d\mathbf{\Gamma} = \int_{\Omega_s^i} f d\mathbf{\Omega}. + \int_{\Gamma_{sq}^i} g\, d\mathbf{\Gamma}. \end{aligned} \quad (23)$$

3.2 Discretization of the weak form

By substitution the RKPI approximation into the LWF equation (23), we obtain

$$\begin{aligned} &\sum_{j=1}^{N} \left(\int_{\Omega_s^i} \phi_j(\mathbf{x}) d\mathbf{\Omega} \right) \hat{T}'_j(t) - \sum_{j=1}^{N} \left[\left(\int_{\Omega_s^i} \mathbf{u}\cdot\boldsymbol{\phi}_{j,k}(\mathbf{x}) d\mathbf{\Omega} \right. \right. \\ &- \int_{\mathrm{L}_s^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} - \int_{\Gamma_{sn}^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} \\ &\left. \left. - \int_{\Gamma_{sd}^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} \right) \hat{T}_j(t) \right] = \int_{\Omega_s^i} f d\mathbf{\Omega} + \int_{\Gamma_{sq}^i} g\, d\mathbf{\Gamma}. \end{aligned} \quad (24)$$

Where $i = 1, 2, 3, ..., N$. $(\cdot)_k$ denotes $\partial(\cdot)_k / \partial x_j$.

Equation (24) can be written in the matrix form as following

$$\mathbf{C}\frac{\partial \mathbf{T}}{\partial t} + \mathbf{D}\mathbf{T} = \mathbf{F}(t), \quad (25)$$

where **C, D** and **F** are matrices described as follows:

$$\mathbf{C} = \left[C_{ij} \right], C_{ij} = \int_{\Omega_s^i} \phi_j(\mathbf{x}) d\mathbf{\Omega} \quad (26)$$

$$\begin{aligned} \mathbf{D} &= \left[D_{ij} \right], D_{ij} \\ &= \int_{\Omega_s^i} \mathbf{u}\cdot\boldsymbol{\phi}_{j.k}(\mathbf{x}) d\mathbf{\Omega} - \int_{\mathrm{L}_s^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} \\ &- \int_{\Gamma_{sn}^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} - \int_{\Gamma_{sd}^i} \mathbf{K}\boldsymbol{\phi}_{j,\mathbf{n}}(\mathbf{x}) d\mathbf{\Gamma} \end{aligned} \quad (27)$$

$$\mathbf{F} = \left[F_i \right], F_i = \int_{\Omega_s^i} f d\mathbf{\Omega} + \int_{\Gamma_{sq}^i} g\, d\mathbf{\Gamma}. \quad (28)$$

3.3 The Time discretization approximation

In the current work, we employ a time-stepping scheme to overcome the time derivative and applying the Crank-Nicolson technique of approximation to equation (25) yields:

$$\left(2\mathbf{C} + \Delta t\, \mathbf{D}\right)\mathbf{T}^{k+1} = \left(2\mathbf{C} + \Delta t\, \mathbf{D}\right)\mathbf{T}^{k} + 2\Delta t\, \mathbf{F}^{\frac{1}{2}} \quad (29)$$

where $\mathbf{T}$ is a matrix described as follows:

$$\mathbf{T} = \left[\hat{\mathbf{T}} \;\; \hat{\mu} \right]^{\mathbf{T}} \quad (30)$$

Assuming that $\hat{T}_i^k$, for $i = 1, 2, ..., N$ and $\hat{\mu}^k$, are known, our aim is to compute $\hat{T}_i^{k+1}$, for $i = 1, 2, ..., N$ and $\hat{\mu}^{k+1}$. Now, we have $N+1$ unknowns, so to compute these unknowns we need one equation, which can be obtained from the non-local boundary condition from equation (23). Substituting trial function equation (1), we obtain:

$$\int_\Omega \left(T^h\right)^{k+1}(\mathbf{x})\,\mathrm{d}\boldsymbol{\Omega} = \int_\Omega \sum_{j=1}^{N} \phi_j(\mathbf{x})\hat{T}_j^{k+1}\mathrm{d}\boldsymbol{\Omega} = \sum_{j=1}^{N}\left[\int_\Omega \phi_j(\mathbf{x})\mathrm{d}\boldsymbol{\Omega}\right]\hat{T}_j^{k+1} = m^{k+1} \quad (31)$$

which can be written in a matrix form as:

$$\mathbf{S}\hat{\mathbf{T}}^{k+1} = m^{k+1}, \quad (32)$$

where $\mathbf{S}$ is a matrix described as follows:

$$\mathbf{S} = \left[S_j\right], S_j = \int_\Omega \phi_j(\mathbf{x})\mathrm{d}\boldsymbol{\Omega}. \quad (33)$$

For nodes on the natural, essential and non-local boundaries, we present an algorithm to impose boundary conditions as follows:

-For node $\mathbf{x}_i \in \boldsymbol{\Gamma}_q$ is on the natural boundary condition,

$$\sum_{j=1}^{N}\left[\int_{\Gamma_{sq}^i} \phi_{j,\mathbf{n}}(\mathbf{x})\mathrm{d}\boldsymbol{\Gamma}\right]\hat{T}_j^{k+1} = \int_{\Gamma_{sq}^i} g\left(\mathbf{x}^l,(k+1)\Delta t\right)\mathrm{d}\boldsymbol{\Gamma} \quad (34)$$

-For node $\mathbf{x}_i \in \boldsymbol{\Gamma}_d$ is on the essential boundary condition,

$$\sum_{j=1}^{N}\left[\int_{\Gamma_{sd}^i} \phi_j(\mathbf{x})\mathrm{d}\boldsymbol{\Gamma}\right]\hat{T}_j^{k+1} = \int_{\Gamma_{sd}^i} h_1\left(\mathbf{x}^l,(k+1)\Delta t\right)\mathrm{d}\boldsymbol{\Gamma} \quad (35)$$

-For node $\mathbf{x}_i \in \boldsymbol{\Gamma}_n$ is on the non-local boundary condition,

$$\sum_{j=1}^{N}\left[\int_{\Gamma_{sn}^i} \phi_j(\mathbf{x})\mathrm{d}\boldsymbol{\Gamma}\right]\hat{T}_j^{k+1} - \int_{\Gamma_{sn}^i} h_0\left(\mathbf{x}^l\right)\mathrm{d}\boldsymbol{\Gamma}\hat{\mu}^{k+1} = 0. \quad (36)$$

IV. NUMERICAL EXPERIMENT

In this section, several cases are considered in a unit square domain given by $0 \le x, y \le 1$. The diffusivity coefficient is 1. The calculations are performed using $r_0 = 0.5h$ and $r_i = 4r_0$. It setting a time-stepping scheme to overcome the time derivative and applying the Crank-Nicolson method of approximation. Tables presented the relative error of method for $T(x,y,t)$ and $\mu(t)$ at time Δt = 0.01, M =3, 6 and 10, N =25, 36 and 81.

Table 1: The relative error for $T(\mathbf{x},t)$.

N/M	3	6	10
25	2.4164e-2	4.6826e-3	2.1780e-3
36	2.1451e-2	3.0671e-3	1.5721e-3
81	7.9962e-3	2.1564e-3	1.3923e-3

Numerical experiment

We suppose that the exact solution of this problems is given by

$$T(\mathbf{x},t) = (1+y)\exp(x+t), \quad 0 \le \mathbf{x} \le 1, \quad t \ge 0,$$
$$\mu(t) = \exp(t).$$

The source term of this problem is:

$$f(\mathbf{x}) = (1+y)\exp(x+t) + u_1(1+y)\exp(x+t) + u_2\exp(x+t) - (1+y)\exp(x+t), \quad 0 \le \mathbf{x} \le 1.$$

The two-dimensional convection-diffusion equation with essential and non-local boundary condition subject to initial.

$$T(\mathbf{x},0) = (1+y)\exp(\mathbf{x}),$$
$$m(t) = \frac{3}{2}\exp(t)\left(\exp(1)-1\right), \quad 0 \le \mathbf{x} \le 1,$$
$$h_0(\mathbf{x}) = \exp(x), \quad h_2(\mathbf{x},t) = 2\exp(x+t),$$
$$h_3(\mathbf{x},t) = (1+y)\exp(t), \quad t \ge 0.$$
$$h_4(\mathbf{x},t) = (1+y)\exp(1+t), \quad t \ge 0,$$
$$h_1(\mathbf{x},t) = h_2(\mathbf{x},t) \cup h_3(\mathbf{x},t) \cup h_4(\mathbf{x},t)$$

RESULTS

The numerical result of the two-dimensional time-dependent convection-diffusion equation with essential and non-local boundary condition subject to initial shows that the relative error of Crank-Nicolson method has been shown to converge by increasing the number of nodal points, which can also increase accuracy. In the same way, accuracy can also be increased by increasing the number of monomial basis. The numerical results of two test problems have been used to verify the efficiency and accuracy of the method.

The relative error for at $T(\mathbf{x},t)$ at M =3, 6 and 10, N = 25, 36 and 81.

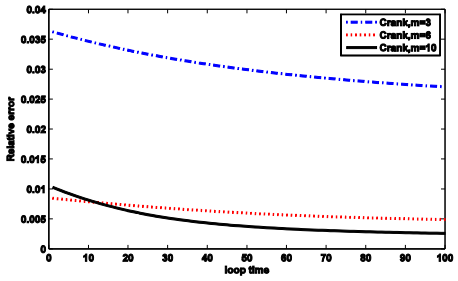


Fig.1.The relative error for $T(\mathbf{x},t)$ at M = 3, 6 and 10 with N = 25.

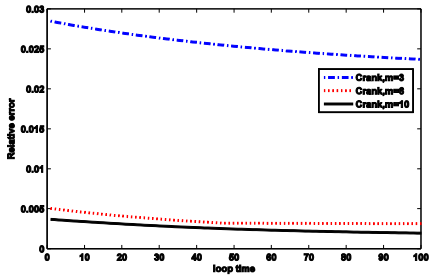


Fig.2. The relative error for $T(\mathbf{x},t)$ at M = 3, 6 and 10 with N = 36.

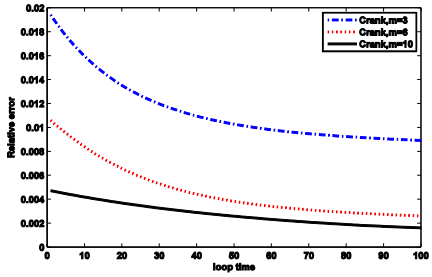


Fig.3. The relative error for $T(\mathbf{x},t)$ at M = 3, 6 and 10 with N = 81.

The relative error for at $\mu(t)$ at$M$ =3, 6 and 10, N = 25, 36 and 81.

Table 2: The relative error for $\mu(t)$.

N/M	3	6	10
25	1.0176e-2	2.1437e-3	1.0176e-3
36	9.3879e-3	1.0592e-3	6.5619e-4
81	8.6656e-3	6.9845e-4	5.2968e-4

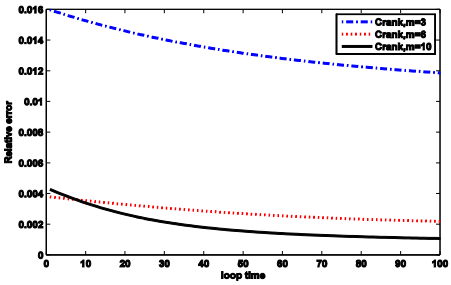


Fig.4. The relative error for $\mu(t)$ at M =3, 6 and 10 with N =25.

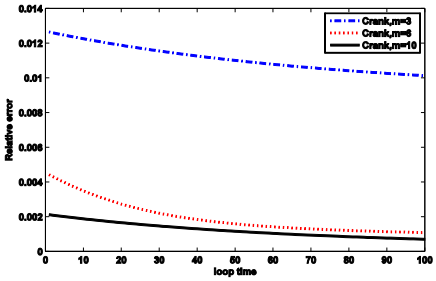


Fig.5. The relative error for $\mu(t)$ at M =3, 6 and 10 with N =36.

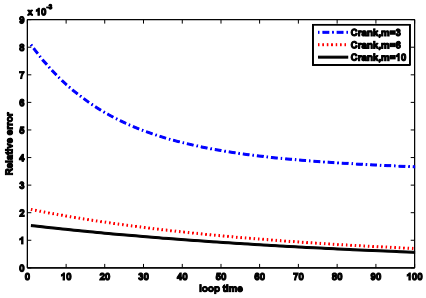


Fig.6. The relative error for $\mu(t)$ at M =3, 6 and 10 with N =81.

CONCLUSION

We have presented the convection-diffusion equation with the non-local boundary conditions arises in many physical phenomena. In this paper, the meshless local Pretrov-Galerkin (MLPG) method based on the reproducing kernel particle interpolation (RKPI) method to solving time-dependent convection-diffusion equation in two-dimensional spaces on a square domain. A technique is proposed to construct shape functions using reproducing kernel function, and the Heaviside step function is used as a test function in each sub-domain to avoid the need for domain integral in local symmetric weak form. Problem is set a time-stepping scheme to overcome the time derivative using the Crank-Nicolson method of approximation. Numerical results revealed that these algorithms have good accuracy. The relative error of Crank-Nicolson method has been shown to converge by increasing the number of nodal points, which can also increase accuracy. In the same way, accuracy can also be increased by increasing the number of monomial basis.The numerical results of two test problems have been used to verify the efficiency and accuracy of the method.

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