ARITHMETIC FUNCTIONS AND THE CROSS - CONVOLUTION

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Abstract - The Cross - convolution product of two arithmetic functions f and g is defined as $(f\nabla g)(n)=\sum_{xy=n}\frac{\chi(n)}{\chi(x)\chi(y)}f(x)g(y)$, where $\chi(n)\neq 0$ for all $n\in N$. We establish properties of arithmetic function under Cross - convolution and compare our results with those of the Dirichlet convolution.

Keywords - Multiplicative function, Completely multiplicative function, Dirichlet convolution, Cross – convolution

I. INTRODUCTION

An arithmetic function is a function $f:Z^+\to C$. Such a function is called *multiplicative* if $f(1)=1$ and $f(mn)=f(m)f(n)$ whenever $\gcd(m,n)=1$. A multiplicative function is called *completely multiplicative*if this equality holds for all $m,n\in N$.

The *"Dirichlet convolution"* or Dirichlet product is a binary operation on a set of arithmetic functions given by

$$(f*g)(n)=\sum_{xy=n}f(x)g(y), \text{ for all } n\in N.$$

Let A be the set of arithmetic functions equipped with addition and multiplication and defined over N, by

$$(f+g)(n)=f(n)+g(n),$$
$$fg(n)=f(n)g(n).$$

The multiplication identity $e\in A$ is defined by $e(n)=1$ for all $n\in N$.

The Dirichlet convolution identity $I\in A$ is defined by $I(1)=1$ and $I(n)=0$ for $n>1$. One note immediately that $(A,+,*,C)$ is a C-algebra under the addition and Dirichlet convolution.

In [1],[3],[7] the authors provide basic properties of arithmetic with respect to the Dirichlet convolution. In [4],[5] the authors derive characterizations of completely multiplicative functions in terms of the Dirichlet convolution.

The convolution $\circ$, also know as binomial convolution, was used by Haukkanen[2] to construct some properties of arithmetic functions f and g is defined as

$$(f\circ g)(n)=\sum_{d|n}\left(\prod_p\binom{v_p(n)}{v_p(d)}\right)f(d)g(n/d),$$

where $\binom{a}{b}$ is the binomial coefficient and $v_p(n)$ is the highest power of p dividing n.

On the other side, we can denote $f\circ g$ as

$$(f\circ g)(n)=\sum_{xy=n}\frac{\xi(n)}{\xi(x)\xi(y)}f(x)g(y),$$

where $\xi(n)=\prod_p v_p(n)!$ for all $n\in N$.

This is obviously $(A,+,\circ,C)$ is a C-algebra under the addition and binomial convolution with the binomial convolution identity I and properties of completely multiplicative functions with respect to the binomial convolution was provided [8].

In purpose of the present study is to generalize the binomial convolution by Cross – convolution, which as follow.

Let $\chi\in A$ be such that $\chi(n)\neq 0$ for all $n\in N$, for $f,g\in A$

$$(f\nabla g)(n)=\sum_{xy=n}\frac{\chi(n)}{\chi(x)\chi(y)}f(x)g(y).$$

It is easy to see that $(A,+,\nabla,C)$ is ring with identity χI. We note that

(1) if χ is completely multiplicative function then $f\nabla g=f*g$

(2) if $\chi=\zeta$ then $f\nabla g=f*g$

(3) if $\chi=\xi$ then $f\nabla g=f\circ g$,

we establish properties of arithmetic function under the Cross - convolution and compare our results with those of the Dirichlet convolution.

II. RESULTS

Throughout this paper we let $\chi\in A$ with $\chi(n)\neq 0$. We first show that $(A,+,*,C)$ is isomorphic to $(A,+,\nabla,C)$.

Theorem 1. The algebra $(A,+,*,C)$ and $(A,+,\nabla,C)$ are isomorphic under the mapping $f\to\frac{f}{\chi}$.

***proof*.** First, consider the mapping $f \to \frac{f}{\chi}$. It is easy to see that this mapping is a bijection on A.
Next, From definition

$$(f\nabla g)(n) = \sum_{xy=n} \frac{\chi(n)}{\chi(x)\chi(y)} f(x)g(y) = \chi(n)(\frac{f}{\chi} * \frac{g}{\chi})(n)$$

for all $n \in N$, we get

$$f\nabla g = \chi(\frac{f}{\chi} * \frac{g}{\chi}) \quad (1)$$

i.e. $\frac{f\nabla g}{\chi} = \frac{f}{\chi} * \frac{g}{\chi}$ and $\frac{f+g}{\chi} = \frac{f}{\chi} + \frac{g}{\chi}$, $\frac{cf}{\chi} = c\frac{f}{\chi}; c \in C$.

This show that $f \to \frac{f}{\chi}$ is an algebra isomorphism. □

We can write the Cross - convolution in term of the Dirichlet convolution.
On the other side, we have

$$f * g = \frac{\chi f \nabla \chi g}{\chi} \quad (2)$$

or $\chi(f * g) = \chi f \nabla \chi g$ for all $f, g, \chi \in A$.

Let f^{-1*} and $f^{-1\nabla}$ denote the inverse f under the Dirichlet convolution $*$ and the Cross- convolution ∇, respectively. They exist if and only if $f(1) \neq 0$.

Theorem 2. For any $f \in A$ with $f(1) \neq 0$,

$$f^{-1*} = \frac{(\chi f)^{-1\nabla}}{\chi} \quad (3)$$

$$f^{-1\nabla} = \chi(\frac{f}{\chi})^{-1*} \quad . \quad (4)$$

***proof*.** Since $f * f^{-1*} = I$ and from (2) we have

$$\frac{\chi f \nabla \chi f^{-1*}}{\chi} = I \text{ or } \chi f \nabla \chi f^{-1*} = \chi I .$$

Thus $\chi f^{-1*} = (\chi f)^{-1\nabla}$ i.e. $f^{-1*} = \frac{(\chi f)^{-1\nabla}}{\chi}$.

On the other side , we have $f \nabla f^{-1\nabla} = \chi I$ and from (1) we obtain

$\chi(\frac{f}{\chi} * \frac{f^{-1\nabla}}{\chi}) = \chi I$ or $(\frac{f}{\chi} * \frac{f^{-1\nabla}}{\chi}) = I$. Hence $\frac{f^{-1\nabla}}{\chi} = (\frac{f}{\chi})^{-1*}$

i.e. $f^{-1\nabla} = \chi(\frac{f}{\chi})^{-1*}$.

The proof is complete. □

Proposition 3. ([8]) If f is multiplicative and $f(p^a) = 0$ for all prime power p^a with $a \geq 2$, then for every $n \geq 1$

$$f^{-1*} = (-1)^{\Omega(n)} \prod_p f(p)^{v_p(n)},$$

where $\Omega(n) = \sum_p v_p(n)$. □

Next, we will generalize Proposition 3. by the Cross-convolution.

Theorem 4. If f, χ are multiplicative and $f(p^a) = 0$ for all prime power p^a with $a \geq 2$, then for every $n \geq 1$

$$f^{-1\nabla}(n) = (-1)^{\Omega(n)} \chi(n) \prod_p \frac{f}{\chi}(p)^{v_p(n)}$$

proof. Since $\frac{f}{\chi}$ is multiplicative,

$\frac{f}{\chi}(p^a) = \frac{f(p^a)}{\chi(p^a)} = 0$ for $a \geq 2$, by (4)

we have $f^{-1\nabla} = \chi(\frac{f}{\chi})^{-1*}$.

By Proposition 3, it can be easily verified that

$$f^{-1\nabla}(n) = (-1)^{\Omega(n)} \chi(n) \prod_p \frac{f}{\chi}(p)^{v_p(n)}. \quad \square$$

Proposition 5. For $w \in A$, $\mu^{-1\nabla} = w$ where w is defined by $w(1) = \alpha(1)^2$ and

$$w(n) = -\chi(1)\chi(n) \sum_{\substack{xy=n \\ y<n}} \frac{\mu}{\chi}(x)\frac{w}{\chi}(y) \text{ if } n > 1 .$$ If χ is multiplicative then $w(n) = 1$ and

$$w(n) = \frac{\chi(n)}{\prod_p \chi^{v_p(n)}(p)} \text{ if } n > 1 . \quad \square$$

Theorem 6. Let f be a multiplicative function. Then f is completely multiplicative if and only if $f(g\nabla h) = fg\nabla fh$ for all $g, h \in A$.

proof. First, we can show that

$$f(g\nabla h)(n) = f(n)\sum_{xy=n} \frac{\chi(n)}{\chi(x)\chi(y)} g(x)h(y)$$

$$= f(n)\chi(n)\sum_{xy=n} \frac{g}{\chi}(x)\frac{h}{\chi}(y)$$

$$= \chi(n)\sum_{xy=n} \frac{fg}{\chi}(x)\frac{fh}{\chi}(y)$$

$$= \chi(n)(\frac{fg}{\chi} * \frac{fh}{\chi})(n)$$

$$= fg\nabla fh(n) .$$

To prove the converse part , we let $g(n) = \mu(n)$, $h(n) = w(n)$. Then we obtains

$\chi I = \chi f I = f\chi I = f(\mu\nabla w) = f\mu\nabla fw$.

Therefore $(f\mu)^{-1\nabla} = fw$.

On the other side, $f\mu$ is multiplicative, $f\mu(p^a) = 0$ for all $a \geq 2$, and according to Theorem 4, we get

$$f(n)w(n) = \chi(n)\prod_p \frac{f}{\chi}(p)^{v_p(n)}$$

$$\frac{fw}{\chi}(n) = \prod_p \frac{f}{\chi}(p)^{v_p(n)} .$$

Hence $\frac{fw}{\chi}(p^a) = \frac{f}{\chi}(p)^a$

$$\frac{f(p^a)}{\chi(p^a)}\frac{\chi(p^a)}{\chi(p)^a} = \frac{f(p)^a}{\chi(p)^a}$$

$f(p^a) = f(p)^a$ for all $a \geq 2$. □

CONCLUSION AND RECOMMENDATIONS

In this paper we obtain some properties of arithmetic functions under the Cross - convolution, Moreover we explore the properties is similar to some properties under the Dirichlet convolution. We will study further properties and extended to several functions in the further.

ACKNOWLEDGEMENTS

This research is partially supported by Department of Mathematics, Faculty of Science and Technology, Valaya Alongkorn Rajabhat University under the Royal Patronage.

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