THE ESTIMATION METHODS OF THE STANDARD WALD DISTRIBUTION PARAMETER

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Abstract - In this paper we study of the performances of the methods for the estimation of the standard Wald distribution parameter. There are three estimation methods as follows: the maximum likelihood method, the adjusted parameter with term $(n-2)/n$ and the adjusted parameter with term $n/(n-2)$. These methods are compared in terms of the absolute bias and the mean square errors. The comparison is based on the Monte Carlo simulation. Results of the simulation studies show that the maximum likelihood method outperforms the other methods where the estimator close to the parameter whereas the adjusted parameter with term $(n-2)/n$ outperforms the other methods where the estimator is greater than the parameter. While, the adjusted parameter with term $n/(n-2)$ outperforms the other methods where the estimator is less than the parameter.

Keywords - Maximum Likelihood Estimate, Parameter Estimation, Standard Wald Distribution.

I. INTRODUCTION

The inverse Gaussian distribution, also known as the Wald distribution, has been applied extensively in many fields such as pedagogy, hydrology, demography, linguistics, employment services and finance[1].Besides, this distribution has proposed a model of lifetime behavior and discusses different reliability features of the distribution [2].The distribution is used to model positively skewed data.The density function of the two-parameter inverse Gaussian distribution, $IG(\mu,\lambda)$, is defined as

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right);\quad x>0,\mu,\lambda>0$$

The standard Wald distribution is a special case of the inverse Gaussian distribution when $\mu=1$. The density function of a parameter standard Wald distribution, $standardWald(\lambda)$, is defined as

$$f(x;\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-1)^2}{2x}\right);\quad x>0,\lambda>0$$

There are many researchers have proposed methods for the estimation of inverse Gaussian distribution parameters such as maximum likelihood estimators, uniformly minimum variance unbiased estimators, moment method, Bayesian method, least squares method, etc.

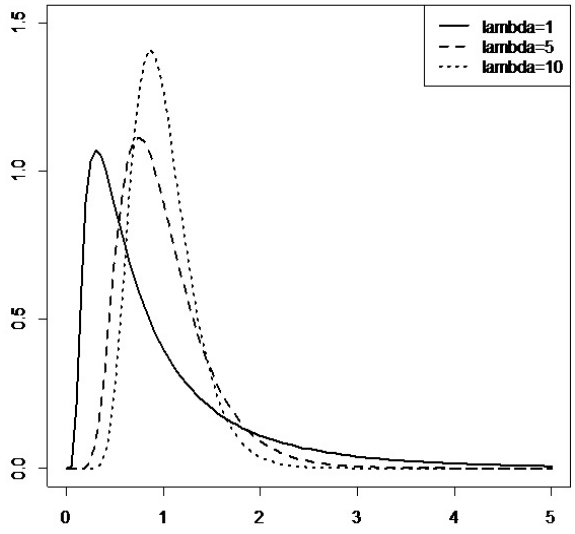

Fig. 1. Standard Wald densities for different λ

Wararit Panichkitkosokul (2008) developed the coefficient of variation and standard deviation estimation methods of inverse Gaussian data.The results indicated that almost sample sizes and parameter values, the absolute bias and the mean square errors of adjusted coefficient of variation and the standard deviation methods with term $(n-2)/n$ were the lowest.Then, Wararit Panichkitkosokul (2010)developed the coefficient of skewness estimation methods of inverse Gaussian distribution with missing values. The results indicated that for all sample sizes and all parameter values, the absolute bias and the mean square errors of adjusted

coefficient of skewness method with term $(n-2)/n$ were the lowest.

In this paper we study the performances of the methods for the estimation of the standard Wald distribution parameter. We consider the maximum likelihood method and the adjusted parameter with two constant terms for the estimation methods of the standard Wald distribution parameter. Simulation results also show the absolute bias and the mean square errors.

This remainder of this paper is organized as follows. Section 2 proposes the estimation methods of the standard Wald distribution parameter. Next, section 3 studies the simulation procedures and the simulation results. Finally, conclusions are presented in Section 4.

II. ESTIMATION OF PARAMETER

2.1. The standard Wald distribution

The probability density function (p.d.f.) of the inverse Gaussian distribution is

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right);\ x>0$$

where $\mu>0$ is the mean parameter and $\lambda>0$ is the scale parameter. The population mean, variance, skewness, and kurtosis of *X* are

$E(X)=\mu$, $Var(X)=\frac{\mu^3}{\lambda}$, $E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right]=3\sqrt{\frac{\mu}{\lambda}}$ and $E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right]=3+\frac{15\mu}{\lambda}$, respectively.

For a random sample $X_1, X_2, ..., X_n$ from inverse Gaussian (μ,λ), the maximum likelihood estimators (MSE) of μ and λ are given by

$$\hat{\mu}=\bar{x} \text{ and } \hat{\lambda}=\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{x}}\right)}.$$

When $\mu=1$ this becomes the standard Wald distribution (λ)

$$f(x;\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-1)^2}{2x}\right);\ x>0$$

where $\lambda>0$ is the scale parameter. The population mean, variance, skewness, and kurtosis of *X* are

$E(X)=1$, $Var(X)=\frac{1}{\lambda}$, $E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right]=\frac{3}{\sqrt{\lambda}}$ and $E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right]=3+\frac{15}{\lambda}$, respectively.

For a random sample $X_1, X_2, ..., X_n$ from standard Wald (λ), the maximum likelihood estimators (MLE) of λ is given by

$$\hat{\lambda}=\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-1\right)}$$

2.2. Estimation methods

In this paper we compare three estimation methods for the parameter λ of the standard Wald distribution. The following estimation methods are considered:

2.2.1. The maximum likelihood method

$$\hat{\lambda}_{MLE}=\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-1\right)}$$

2.2.2. The adjusted parameter with term $(n-2)/n$

$$\hat{\lambda}_{adj1}=\left(\frac{n-2}{n}\right)\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-1\right)}\right)=\left(\frac{n-2}{n}\right)\hat{\lambda}_{MLE}$$

2.2.3. The adjusted parameter with term $n/(n-2)$

$$\hat{\lambda}_{adj2}=\left(\frac{n}{n-2}\right)\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-1\right)}\right)=\left(\frac{n}{n-2}\right)\hat{\lambda}_{MLE}$$

III. SIMULATION STUDY

3.1. Simulation Procedures

In the simulation study, we generate the random samples from the standard Wald distribution in R version 3.4.2. The simulation is as follows:

3.1.1. consider sample sizes $n = 5,10,20,\ 30,50$ and parameter $\lambda = 0.1,0.5,1,3,5,10,20$.

3.1.2. generate by the Monte Carlo simulation $M = 50{,}000$ random samples from the standard Wald distribution. For each method under consideration, we obtain 50,000 estimates $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{50{,}000}$ of theparameter λ. Then, compute the sample means $\hat{\lambda}$

$$\hat{\lambda} = \frac{\sum_{i=1}^{50{,}000} \hat{\lambda}_i}{50{,}000}$$

3.1.3. compute the absolute bias $\left(\left|Bias\right|\right)$ and the mean square errors $\left(MSE\right)$ of three estimation methods. $\left|Bias\right|$ and MSE are given by

$$\left|Bias\right| = \left| \frac{\sum_{i=1}^{50{,}000} \hat{\lambda}}{50{,}000} - \lambda \right| \quad \text{and} \quad MSE = \frac{\sum_{i=1}^{50{,}000} \left(\lambda - \hat{\lambda}\right)^2}{50{,}000}$$

3.1.4. compare the $\left|Bias\right|$ and the MSE of the estimation methods. The estimators with the lowest $\left|Bias\right|$ and MSE are preferred.

3.2. Simulation Results

The simulation results show that the maximum likelihood method, $\hat{\lambda}_{MLE}$, outperforms the other methods where the $\hat{\lambda}_{MLE} \approx \lambda$ whereas the adjusted parameter with term $(n-2)/n$, $\hat{\lambda}_{adj1}$, outperforms the other methods where the $\hat{\lambda}_{MLE} > \lambda$. Meanwhile, the adjusted parameter with term $n/(n-2)$, $\hat{\lambda}_{adj2}$, outperforms the other methods where the $\hat{\lambda}_{MLE} < \lambda$. Furthermore, the results of the comparison of the $\left|Bias\right|$ and the MSE for all situations in this study show that for the $\hat{\lambda}_{MLE} \approx \lambda$, the $\left|Bias\right|$ and the MSE of the $\hat{\lambda}_{MLE}$ are the lowest. For the $\hat{\lambda}_{MLE} > \lambda$, the $\left|Bias\right|$ and the MSE of the $\hat{\lambda}_{adj1}$ are the lowest. For the $\hat{\lambda}_{MLE} < \lambda$,the $\left|Bias\right|$ and the MSE of the $\hat{\lambda}_{adj2}$ are the lowest.This simulation results are shown in **Table 1**.

3.3. Discussions

In this study we assumed that the parameter is known, but the parameter is generally unknown**.** So, we would suggest that analyze the goodness of fit test to check the distribution fit before estimate the parameter. Besides, we should study the best performance of the methods for the estimation of the standard Wald distribution parameter for various situations clearly for future research**.**

CONCLUSIONS

The performances of three estimation methods of the standard Wald distribution parameter were compared in this study. The maximum likelihood method showed superiority over the other methods where the estimator close to the parameter. The adjusted parameter with term $(n-2)/n$showed superiority over the other methods where the estimator is greater than the parameter.The adjusted parameter with term $n/(n-2)$showed superiority over the other methods where the estimator is less than the parameter.

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n	λ	Estimators			*Bias*			*MSE*		
		$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{adj1}$	$\hat{\lambda}_{adj2}$	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{adj1}$	$\hat{\lambda}_{adj2}$	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{adj1}$	$\hat{\lambda}_{adj2}$
5	0.1	0.038681	0.023209	**0.064469**	0.061319	0.076791	**0.035531**	7.52E-08	1.18E-07	**2.52E-08**
	0.5	0.726429	**0.435858**	1.210716	0.226429	**0.064142**	0.710716	1.03E-06	**8.23E-08**	1.01E-05
	1	0.611628	0.366977	**1.019380**	0.388372	0.633023	**0.019380**	3.02E-06	8.01E-06	**7.51E-09**
	3	4.313740	**2.588244**	7.189567	1.313740	**0.411756**	4.189567	3.45E-05	**3.39E-06**	0.000351
	5	**5.305892**	3.183535	8.843153	**0.305892**	1.816465	3.843153	**1.87E-06**	6.60E-05	0.000295
	10	16.573340	**9.944002**	27.622230	6.573336	**0.055998**	17.622230	0.000864	**6.27E-08**	0.006211
	20	12.778530	7.667121	**21.297560**	7.221466	12.332880	**1.297557**	0.001043	0.003042	**3.37E-05**
10	0.1	0.181623	**0.145298**	0.227029	0.081623	**0.045298**	0.127029	1.33E-07	**4.10E-08**	3.23E-07
	0.5	**0.481687**	0.385350	0.602109	**0.018313**	0.114650	0.102109	**6.71E-09**	2.63E-07	2.09E-07
	1	1.536408	**1.229126**	1.920510	0.536408	**0.229126**	0.920510	5.75E-06	**1.05E-06**	1.69E-05
	3	**3.000898**	2.400718	3.751122	**0.000898**	0.599282	0.751122	**1.61E-11**	7.18E-06	1.13E-05
	5	2.597642	2.078114	**3.247053**	2.402358	2.921886	**1.752947**	0.000115	0.000171	**6.15E-05**
	10	7.477474	5.981979	**9.346842**	2.522526	4.018021	**0.653158**	0.000127	0.000323	**8.53E-06**
	20	27.782890	**22.226310**	34.728610	7.782886	**2.226308**	14.728610	0.001211	**9.91E-05**	0.004339
20	0.1	0.081503	0.073353	**0.090559**	0.018497	0.026647	**0.009441**	6.84E-09	1.42E-08	**1.78E-09**
	0.5	0.401921	0.361729	**0.446579**	0.098079	0.138271	**0.053421**	1.92E-07	3.82E-07	**5.71E-08**
	1	1.299879	**1.169891**	1.444310	0.299879	**0.169891**	0.444310	1.80E-06	**5.77E-07**	3.95E-06
	3	2.799401	2.519461	**3.110446**	0.200599	0.480539	**0.110446**	8.05E-07	4.62E-06	**2.44E-07**
	5	6.392206	**5.752985**	7.102451	1.392206	**0.752985**	2.102451	3.88E-05	**1.13E-05**	8.84E-05
	10	**9.963707**	8.967336	11.070790	**0.036293**	1.032664	1.070786	**2.63E-08**	2.13E-05	2.29E-05
	20	11.711710	10.540540	**13.013010**	8.288288	9.459459	**6.986987**	1.37E-03	1.79E-03	**9.76E-04**
30	0.1	0.104654	**0.097677**	0.112129	0.004654	**0.002323**	0.012129	4.33E-10	**1.08E-10**	2.94E-09
	0.5	0.564706	**0.527059**	0.605042	0.064706	**0.027059**	0.105042	8.37E-08	**1.46E-08**	2.21E-07
	1	0.946735	0.883619	**1.014359**	0.053265	0.116381	**0.014359**	5.67E-08	2.71E-07	**4.12E-09**
	3	**2.986496**	2.787396	3.199817	**0.013504**	0.212604	0.199817	**3.65E-09**	9.04E-07	7.99E-07
	5	3.369038	3.144435	**3.609684**	1.630962	1.855565	**1.390316**	5.32E-05	6.89E-05	**3.87E-05**
	10	11.185930	**10.440200**	11.984930	1.185933	**0.440204**	1.984928	2.81E-05	**3.88E-06**	7.88E-05
	20	16.568070	15.463530	**17.751500**	3.431934	4.536472	**2.248501**	0.000236	0.000412	**0.000101**
50	0.1	0.091683	0.088016	**0.095503**	0.008317	0.011984	**0.004497**	1.38E-09	2.87E-09	**4.04E-10**
	0.5	0.473370	0.454436	**0.493094**	0.026630	0.045564	**0.006906**	1.42E-08	4.15E-08	**9.54E-10**
	1	0.929177	0.892010	**0.967893**	0.070823	0.107990	**0.032107**	1.00E-07	2.33E-07	**2.06E-08**
	3	4.206087	**4.037843**	4.381340	1.206087	**1.037843**	1.381340	2.91E-05	**2.15E-05**	3.82E-05
	5	5.254720	**5.044531**	5.473666	0.254720	**0.044531**	0.473666	1.30E-06	**3.97E-08**	4.49E-06
	10	10.421200	**10.004360**	10.855420	0.421204	**0.004356**	0.855421	3.55E-06	**3.79E-10**	1.46E-05
	20	25.532380	**24.511090**	26.596230	5.532381	**4.511086**	6.596231	0.000612	**0.000407**	0.000870

Table 1: The simulation results of the estimation methods

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