UNSTEADY FLOW OF AN INCLINED SYMMETRIC THIN LIQUID FILMS

JOSEPH G. ABDULAHAD

[1]Mathematics Department – Zakho university - Iraq
Email: [1]xxxxxx@gmail.com

Abstract - In this paper we construct a mathematical model to describe the unsteady incompressible flow in an inclined symmetric double sided thin liquid films in two dimensions. We apply the Navier – Stokes equations with an appropriate boundary conditions to obtain the non – linear differential equation of the film thickness.

Keywords - Thin Liquid Films, Navier-Stokes Equations, dimensional Analysis.

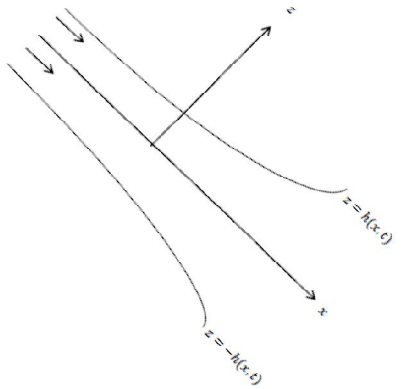
I. INTRODUCTION

The nature of physical surfaces has been studied over a long period of time. The macroscopic and molecular properties of the surface were investigated by physical, chemical and biological scientists, while the mathematicians have been concerned with mathematical problems, especially with the solution of the boundary value problems which arise from a macroscopic representation of the molecular properties of the surface. The mechanics of free surface liquid film is important in many applications. [1] considered the steady flow of a horizontal symmetric thin liquid film, the differential equations that governs such flow is derived and solved numerically. A mathematical model was considered by [2] to describe the unsteady flow in a horizontal thin liquid films and conclude that the film thickness of some liquids namely glycerin and olive oil decreases as the time increases. [3]investigated the nature of the boundary conditions at the free surface of the liquid, when the liquid is in the form of a moving thin filmand developed a theory of steady motion in a single semi–infinite film with zero shear-stress surfaces rather than the no slip condition as Mysels[7] suggested that the surface of the film behaves as a rigid surface.. A mathematical model is constructed by [4] to describe the thinning process of an inclined thin liquid film over a solid surfacethat drainages under the action of gravity.The stability analysis of thin liquid films are investigated by [5]by using the long wave method. [6] investigated the dynamics ofan inclined thin liquid filmsof variable thickness for steady and unsteady flow and when the film is stationary and uniform.

The purpose of this work is to study the unsteady flow of an inclined symmetric double-sided thin liquid film. The governing non-linear differential equation is obtained analytically and the solution for the thicknessof the liquid film equation is determined numerically. The similarity method is used to isolate the time dependence and then the shape of the film will depend on one variable only.

II. FORMULATION AND GOVERNING

To describe the flow of a viscous fluid within a double sided inclined symmetric thin film in two dimensions, we take the Cartesian coordinates x and z in which the x axis is the axis of symmetry, and the flow is predominantly in the x directions and the z axis is perpendicular to the plane of the film. In figure 2.1 the transverse dimensions are greatly enlarged by comparison with the longitudinal dimensions for clarity of presentation.



Figure(2.1): Physical model for symmetric thin liquid film

Let the equations of the bounding surfaces of the film be

$$z = \pm h(x,t) \qquad (2.1)$$

And $\frac{\partial h}{\partial x} << 1$ for all x and t.

In differential form, the Navier-Stokes equations in x and z directions respectively are

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \rho g\sin\theta \qquad (2.2)$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] + \rho g\sin\theta \qquad (2.3)$$

Let the velocity distribution be defined as

$$u = u(x,t)$$
$$v = 0 \quad (2.4)$$
$$w = -z\frac{\partial u(x,t)}{\partial x}$$

And the pressure is a function of x and t only.

The distribution (2.4) satisfies the continuity equation for incompressible flow which has the form

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad (2.5)$$

Since $z = h(x,t)$is the equation of the free surface of the liquid film, then the conservation of mass across the film thickness has the form:

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0 \qquad (2.6)$$

Since the slop of the liquid film is so small,the distribution satisfies the following boundary conditions:

The tangential stress condition

$$\tau = \mu\left(\frac{\partial u}{\partial z}\right)_s = 0 \qquad (2.7)$$

Where the subscript s denotes values at the bounding surface of the liquid film.

And the normal stress condition at the surface is given by

$$\omega = \left(-p+2\mu\frac{\partial u}{\partial z}\right)_s \qquad (2.8)$$

From the continuity equation (2.5), equation (2.8) gives

$$\omega = \left(-p-2\mu\frac{\partial u}{\partial x}\right)_s \qquad (2.9)$$

Where σ is the surface tension of the liquid film and since the slop of the film is so small the curvature k of the film reduces to give

$$k = \frac{\partial^2 h}{\partial x^2} \qquad (2.10)$$

Also , on the surface of the film , the normal component of stress is given by

$$\omega = \sigma\frac{\partial^2 h}{\partial x^2} \qquad (2.11)$$

Now from equations (2.9) and (2.11) and on the surface of the film, we get

$$p = -2\mu\frac{\partial u}{\partial x}-\sigma\frac{\partial^2 h}{\partial x^2} \qquad (2.12)$$

Equation (2.12) is valid everywhere.

Differentiate equation (2.12) with respect to x, we have

$$\frac{\partial p}{\partial x} = -2\mu\frac{\partial^2 u}{\partial x^2}-\sigma\frac{\partial^3 h}{\partial x^3} \qquad (2.13)$$

From (2.4), the longitudinal equation of motion in x direction (2.2) reduces to give

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right] = -\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2} + \rho g\sin\theta \qquad (2.14)$$

From equations (2.13) and (2.14), we have

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right] = 3\mu\frac{\partial^2 u}{\partial x^2}+\sigma\frac{\partial^3 h}{\partial x^3}+\rho g\sin\theta \qquad (2.15)$$

Again from the transverse equation of motion (2.3) and from the velocity distribution (2.4), we obtain

$$\frac{\partial p}{\partial z} = \left[\rho u\frac{\partial^2 u}{\partial x\,\partial t}+\rho u\frac{\partial^2 u}{\partial x^2}-\rho\left(\frac{\partial u}{\partial x}\right)^2-\mu\frac{\partial^3 u}{\partial x^3} - \rho g\cos\theta\right]z \qquad (2.16)$$

Integrating both sides of equation (2.16) with respect to z, we get

$$p = \left[\rho u\frac{\partial^2 u}{\partial x\,\partial t}+\rho u\frac{\partial^2 u}{\partial x^2}-\rho\left(\frac{\partial u}{\partial x}\right)^2-\mu\frac{\partial^3 u}{\partial x^3} - \rho g\cos\theta\right]\frac{z^2}{2}+f(x,t) \qquad (2.17)$$

That is

$$p = p(x,t)+O(z), |z|\le h \qquad (2.18)$$

Thus equations (2.6) and (2.15) are the governing equations of the inclined thin liquid films for unsteady flow.

III. FLOW WITH NEGLIGIBLE INERTIA

We now consider unsteady flow of and within liquid film but with negligible inertia and thus the longitudinal equation ofmotion (2.15) reduces to give

$$3\mu\frac{\partial^2 u}{\partial x^2}+\sigma\frac{\partial^3 h}{\partial x^3}+\rho g\sin\theta = 0 \qquad (3.1)$$

Integrating both sides of equation (3.1) twice with respect to x, we get

$$u = \frac{1}{3\mu}\left(G(t)x+F(t)-\sigma\frac{\partial h}{\partial x} - (\rho g\sin\theta)\frac{x^2}{2}\right) \qquad (3.2)$$

Where G(t) and F(t) are arbitrary functions of t and can be determined from asymptotic or initial conditions.

Case1: If $G(t)\neq 0$, we write$H(t)x = G(t)x+F(t)$.

From equation (2.6), equation (3.2) after some simplifications becomes

$$3\mu\frac{\partial h}{\partial t}+H(t)x\frac{\partial h}{\partial x}-\frac{1}{2}\rho g\sin\theta x^2\frac{\partial h}{\partial x}-\sigma\left(\frac{\partial h}{\partial x}\right)^2 + H(t)h-\rho g\sin\theta xh-\sigma\frac{\partial^2 h}{\partial x^2}h = 0 \qquad (3.3)$$

Case2: If $G(t)=0$, we have

$$3\mu\frac{\partial h}{\partial t}+F(t)\frac{\partial h}{\partial x}-\rho g\sin\theta x^2\frac{\partial h}{\partial x}-\sigma\left(\frac{\partial h}{\partial x}\right)^2-\rho g\sin\theta xh-\sigma\frac{\partial^2 h}{\partial x^2}h=0 \qquad (3.4)$$

IV. SIMILARITY SOLUTIONAND NON-DIMENSIONAL ANALYSIS

For case1, let us suppose that the length scale associated with x and h vary with t in a different way. For non-trivial solution of equation (3.3, we introduce the non dimensional variables as follows:

$$x=\frac{\mu}{\rho g t}\varepsilon$$

$$h(x,t)=\frac{\mu^3}{\rho^2 g^2\sigma t^3}f(\varepsilon)$$

$$H(t)=\frac{\mu}{t}$$

Thus equation (3.3) in non-dimensional form becomes

$$f(\varepsilon)\frac{\partial^2 f}{\partial\varepsilon^2}+\left(\frac{\partial f}{\partial\varepsilon}\right)^2+\sin\theta\,\varepsilon f(\varepsilon)+\frac{1}{2}\sin\theta\,\varepsilon^2\frac{\partial f}{\partial\varepsilon}-4\varepsilon\frac{\partial f}{\partial\varepsilon}+8f(\varepsilon)=0 \qquad (3.5)$$

For case2, when $G(t)=0$, we define thefollowing non-dimensional variables:

$$x=\frac{\mu}{\rho g t}\varepsilon$$

$$h(x,t)=\frac{\mu^3}{\rho^2 g^2\sigma t^3}f(\varepsilon)$$

$$F(t)=\frac{\mu^2}{\rho g t^2}$$

And thus equation (3.4) in dimensionless form has the form

$$f(\varepsilon)\frac{\partial^2 f}{\partial\varepsilon^2}+\left(\frac{\partial f}{\partial\varepsilon}\right)^2+\sin\theta\,\varepsilon f(\varepsilon)+\frac{1}{2}\sin\theta\,\varepsilon^2\frac{\partial f}{\partial\varepsilon}-(3\varepsilon+1)*\frac{\partial f}{\partial\varepsilon}+9f(\varepsilon)=0 \qquad (3.6)$$

Let$f(\varepsilon)=\varepsilon^{-n}$, where $\varepsilon\to\infty$

Substituting $f(\varepsilon)$and its derivative in equation (3.5), we get

$$n(n+1)\varepsilon^{-n-1}+n^2\varepsilon^{-n-1}+\sin\theta-\frac{1}{2}\sin\theta\,n+4n\varepsilon^{-1}+8\varepsilon^{-1}=0 \qquad (3.7)$$

and then in equation (3.6)and by taking the$\lim_{\eta\to\infty}$, we get $n=2$. That is the solutions of the equations for both cases are asymptotic to $\frac{1}{\varepsilon^2}$ as $\varepsilon\to\infty$.

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