CONVERGENCE RATE FOR THE METHOD OF FINDING THE LARGEST EIGEN VALUE OF NONNEGATIVE WEAKLY IRREDUCIBLE POLYNOMIALS

[1]NURFADHILAH IBRAHIMA, [2]AZWANI ALIASB

[1,2]School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, Kuala Nerus, Malaysia
E-mail: [1]afadhilah@umt.edu.my, [2]bazwani.alias@umt.edu.my

Abstract - Eigenvalue problem can be found in many fields such as aquatic conservation, ship structure and marine energy system. In 2014, an algorithm for finding the largest eigenvalue of nonnegative polynomials was introduced by Ibrahim. The method has been proven to be convergent but its rate of convergence has not been discussed in literature. In this paper we will show that the rate of convergence for the algorithm is linear under weak irreducibility condition.

Index Terms - Convergence rate, Iterative method, Eigenvalue, Polynomial.

I. INTRODUCTION

Eigenvalue is widely used in many applications including engineering, quantum physics, dynamical system, nonlinear wave, theory of graph and in others disciplines. In engineering, it can be found in principal stresses in mechanics. In dynamic system it is used for analyzing the stability. In structural engineering eigenvalues are used to determine the response of structures during earthquake or wind-storm. As a matter of fact, Google search is also an eigenvalue problem.

In nonlinear wave, solitary waves or wave packets can be modeled mathematically in the nonnegative coefficient in the terms of modified Korteweg-de Vries equation and Ostrovsky equation respectively when the real eigenvalue is obtained through discrete spectrum [1,2].

In this paper, we will consider the eigenvalue of the following polynomial.
Let $: R^n \to R^n$,

$$P(x) = \sum_{\alpha \in R_+^n} a_\alpha x^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $a_\alpha = (a_{\alpha_1}, \cdots, a_{\alpha_n})$ and $x^\alpha = (x_1^{\alpha_1}, \cdots, x_n^{\alpha_n})$, be an irreducible polynomial with nonnegative coefficients.

We call a number $\lambda \in C$ an eigenvalue of P and a nonzero vector $x \in C$ the associated eigenvector if

$$P_i(x) = \lambda x_i^\delta, \quad i = 1, \dots, n.$$

In 2014, an algorithm for finding the largest eigenvalue of nonnegative polynomials was introduced by Ibrahim [3]. The method has been proven to be convergent for irreducible polynomial with nonnegative coefficients but its rate of convergence has not been discussed in literature. In this paper we will show that the method is also convergent for weakly irreducible nonnegative polynomial. Then we will prove that the rate of convergence for the algorithm is linear under weak irreducibility condition.

II. ALGORITHM

The following is the algorithm that was in [3] for finding the largest eigenvalue of nonnegative irreducible polynomials. Largest eigenvalue is also known as spectral radius.

Algorithm 1 [3]
Step 0 : Choose $x^{(1)} \in \mathbb{R}^n_{>0}$. Let $Q(x) = P(x) + x^{[\delta]}$ and let $k = 1$.

Step 1 : Compute

$$Q_i(x^{(k)}) = \sum_{\alpha \in \mathbb{R}^n_+} a_{i\alpha} x^\alpha + x^{[\delta]}, \; i = 1, 2, \dots, n,$$

$$\underline{\mu}_k = \min_{i \in \{1,2,\dots\}, x_i^{(k)} > 0} \frac{Q_i(x^{(k)})}{(x_i^{(k)})^\delta},$$

$$\overline{\mu}_k = \max_{x_i^{(k)} > 0} \frac{Q_i(x^{(k)})}{(x_i^{(k)})^\delta},$$

Step 2 : If $\overline{\mu}_k = \underline{\mu}_k$, then let $\mu = \overline{\mu}_k$ and stop. Otherwise, compute

$$x^{(k+1)} = \frac{(Q(x^{(k)}))^{\left[\frac{1}{\delta}\right]}}{\left\| Q(x^{(k)}))^{\left[\frac{1}{\delta}\right]} \right\|_p},$$

Replace k by $k + 1$ and go to step 1.

III. WEAKER CONVERGENT CONDITION

First we show that Algorithm 1 has weaker convergent condition. In order to do that, we need the following existing results.

Corollary 1 [3]. Let $= (P_1, \dots, P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with P_i is a polynomial of degree $d_i \geq 1$ and the coefficient of each monomial in P_i is nonnegative. Let $\delta = max(d_1, \dots, d_n)$. Suppose that $Q(x) = P(x) + x^{[\delta]}$. Then $\lambda + 1$ is an eigenvalue of $Q(x)$ if and only if λ is an eigenvalue of $P(x)$. In this case, they have the same eigenvector. If P is weakly irreducible, then Q is weakly primitive.

Corollary 2 [3]. Let $P = (P_1, \dots, P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map, where each P_i is a polynomial of degree $d_i \geq 1$ with a nonnegative coefficients. Let $\delta = max(d_1, \dots, d_n)$ and $\|x\|_p$ be the ℓ_p-norm where $\|x\|_p = a$ for $a, p > 0$. Assume that P is weakly irreducible. Then, the unique scalar λ such that there is a positive vector $u$with $P_i(u) = \lambda u_i^{d_i}$ for all $i \in \{1,2, \dots, n\}$ satisfies

$$\lambda = \inf_{x \in \mathbb{R}^n_{>0}, \|x\|_p = a} \max_{i \in \{1,2,\dots,n\}} \frac{P_i(x)}{x_i^{\delta}},$$

$$= \sup_{x \in \mathbb{R}^n_{>0}, \|x\|_p = a} \min_{i \in \{1,2,\dots,n\}, x_i \neq 0} \frac{P_i(x)}{x_i^{\delta}},$$

Now we can prove the weaker convergent condition for Algorithm 1.

Theorem 1. Let P be as in Corollary 2, and P is weakly irreducible nonnegative polynomial. Then Algorithm 1 produce the value μ_0 that satisfies $Q(x) = \mu_0 x^{[\delta]}$ in finite number of steps or generate two convergent sequences $\{\underline{\mu}_k\}$ and $\{\overline{\mu}_k\}$, both of which converge to μ_0. The largest eigenvalue of P is $\mu_0 - 1$.

Proof. By Corollary 1, Q is weakly primitive and by Corollary 5.1 [4], Algorithm 1 converges when P is weakly irreducible nonnegative polynomial.

IV. RATE OF CONVERGENT

In this section, we will show that Algorithm 1 has linear convergence when A is a nonnegative irreducible polynomial. We use the same argument as in [5].
Define

$$G(z) = \beta(z) = P(x) + x^{[\delta]}$$

$$D(z) = G(z)^{\left[\frac{1}{M-1}\right]},$$

$$\zeta(z) = \frac{D(z)}{\phi(D(z))},$$

Where $\phi: R_+^N \to R_+$ is defined by

$$\phi(z) = \|z\|_1 = \sum_{i=1}^{N} z_i,$$

for any nonnegative $z \in R_+^N$. We can see that the sequence $\{z^{(k)}\}$ in Algorithm 1 is generated by

$$z^{(k+1)} = \zeta(z^{(k)}), k = 1,2, \dots, \quad (1)$$

And $\phi(z^{(k)}) = 1$ for all $= 1,2, \dots$.

Theorem 2 (Chapter 2,[6]). If A is an irreducible nonnegative square matrix, then
(i) the spectral radius $\rho(A)$ is an eigenvalue;
(ii) there exists a nonnegative vector $x_0 > 0$, i.e. all components of x_0 are positive, such that $Ax_0 = \rho(A)x_0$;
(iii) (uniqueness) if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(A)$;
(iv) $\rho(A)$ is a simple eigenvalue of A;
(v) if λ is an eigenvalue of A, then $|\lambda| \leq \rho(A)$. Furthermore, if a nonnegative matrix A is primitive, then $\rho(A) > |\lambda|, \forall \lambda \in \sigma(A) \backslash \{\rho(A)\}$, where $\sigma(A)$ is the spectrum of A.

Proposition 1 [7,8]. The spectral radius of an $n \times n$ matrix A is characterized by the equality

$$\rho(A) = \inf_{||.|| \in N} ||A||$$

where N denotes the set of all possible spectral norms of A. For any $\epsilon > 0$, there exists a spectral norm $||.|| \in N$ such that $||A|| \leq \rho(A) + \epsilon$.

Lemma 1 . Let A, u_0, x_0 and y_0 be as in Theorem 1 and let $\zeta'(z_0)$ be the Jacobian of the function ζ at z_0.Then, $\rho(\zeta'(z_0)) < 1$.
Proof. Let u_0 be the largest singular value of β and z_0 be the corresponding eigenvector. We have $\zeta(z_0) = \Gamma(z_0)/\phi(\Gamma(z_0))$.We want to know

$$\rho(\zeta'(z_0)) = \rho\left(\frac{\Gamma'(z_0)\phi(\Gamma(z_0)) - \Gamma(z_0)\phi'(\Gamma(z_0))}{\phi^2(\Gamma(z_0))}\right) < 1.$$

We have $G(z_0) = \beta(z_0) = u_0 z_0^{[M-1]}$ and $\phi(z_0) = 1$. Hence, $\Gamma(z_0) = (G(z_0))^{\left[\frac{1}{M-1}\right]} = u_0^{\left[\frac{1}{M-1}\right]} z_0$. Let $u_1 = u_0^{\left[\frac{1}{M-1}\right]}$ To we have

$$\Gamma(z_0) = u_1 z_0.$$

Now we compute $\Gamma'(z_0)$, the Jacobian of Γ at z_0. Let

$$\Gamma(z_0) = (G(z_0))^{\left[\frac{1}{M-1}\right]} = \begin{bmatrix} (G_1(z_0))^{\left[\frac{1}{M-1}\right]} \\ (G_2(z_0))^{\left[\frac{1}{M-1}\right]} \\ \vdots \\ (G_N(z_0))^{\left[\frac{1}{M-1}\right]} \end{bmatrix},$$

$$\nabla\left((G_1(z_0))^{\left[\frac{1}{M-1}\right]}\right) = \frac{1}{M-1}(G_1(z_0))^{\left[\frac{2-M}{M-1}\right]} \nabla G_1(z_0).$$

By the same method, we can get

$$\nabla\left(\left(G_i(z_0)\right)^{\left[\frac{1}{M-1}\right]}\right) = \frac{1}{M-1}\left(G_i(z_0)\right)^{\left[\frac{2-M}{M-1}\right]}\nabla G_i(z_0) \text{for} i = 1,\dots,N.$$

Thus the Jacobian of Γat z_0,

$$\Gamma'(z_0) = \nabla\left(G(z_0)\right)^{\left[\frac{1}{M-1}\right]}$$

$$= \begin{bmatrix} \nabla\left(G_1(z_0)\right)^{\left[\frac{1}{M-1}\right]} \\ \nabla\left(G_2(z_0)\right)^{\left[\frac{1}{M-1}\right]} \\ \vdots \\ \nabla\left(G_N(z_0)\right)^{\left[\frac{1}{M-1}\right]} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{M-1}\left(G_1(z_0)\right)^{\left[\frac{2-M}{M-1}\right]}\nabla G_1(z_0) \\ \frac{1}{M-1}\left(G_2(z_0)\right)^{\left[\frac{2-M}{M-1}\right]}\nabla G_2(z_0) \\ \vdots \\ \frac{1}{M-1}\left(G_N(z_0)\right)^{\left[\frac{2-M}{M-1}\right]}\nabla G_N(z_0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{M-1}\left(G_1(z_0)\right)^{\left[\frac{2-M}{M-1}\right]} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{M-1}\left(G_N(z_0)\right)^{\left[\frac{2-M}{M-1}\right]} \end{bmatrix} \times$$

$$\begin{bmatrix} \nabla G_1(z_0) \\ \nabla G_2(z_0) \\ \vdots \\ \nabla G_N(z_0) \end{bmatrix}$$

$$= diag\left(\frac{1}{M-1}\left(G(z_06)\right)^{\left[\frac{2-M}{M-1}\right]}\right) G'(z_0)$$

$$= \frac{1}{M-1}\, diag\left((u_1 z_0)^{[2-M]}\right) G'(z_0),$$

where $\frac{1}{M-1}\, diag\left((\mu_1 z_0)^{[2-M]}\right)$ is a constant with $\mu_1 > 0$ and z_0 is a positive vector. Thus $\mathcal{G}\left(\Gamma'(z_0)\right) = \mathcal{G}\left(G'(z_0)\right)$. By Corollary 3, $\Gamma'(z_0)$ is a primitive matrix. Since $\Gamma'(z_0)$ is a primitive matrix, by Theorem 2, the eigenvalues $v_1, v_2, \dots, v_N$ of $\Gamma'(z_0)$ can be ordered as follows:

$$v_1 = \rho\left(\Gamma'(z_0)\right) > |v_2| \geq |v_3| \geq \cdots \geq |v_N|.$$

For all $t > 1$, we expand $\Gamma(tz_0)$ about z_0 using Taylor's Series,

$$tu_1 z_0 = \Gamma(tz_0)$$
$$= \Gamma(z_0) + \Gamma'(z_0)(tz_0 - z_0) + 0(\|tz_0 - z_0\|)$$
$$= u_1 z_0 + (t-1)\Gamma'(z_0)z_0 + 0(t-1)$$
$$(t-1)u_1 z_0 = (t-1)\Gamma'(z_0)z_0 + 0(t-1),$$

which implies $\Gamma'(z_0)z_0 = u_1 z_0$. Since $\Gamma'(z_0)$ is a primitive matrix and $z_0 > 0$, by Theorem 2, z_0 is an eigenvector of $\Gamma'(z_0)$ associated with the largest eigenvalue, $u_1 = v_1$. Hence, $\phi\left(\Gamma(z_0)\right) = \phi(\mu_1 z_0) = u_1$.

We also have $\phi\left(\Gamma(z_0)\right) = \Gamma_1(z_0) + \Gamma_2(z_0) + \cdots + \Gamma_N(z_0)$, and $\phi'\left(\Gamma(z_0)\right) = \Gamma_1'(z_0) + \Gamma_2'(z_0) + \cdots + \Gamma_N'(z_0) = e\Gamma'(z_0)$, where e is the row vector of all the ones with N dimension. Thus,

$$\zeta'(z_0) = \frac{\Gamma'(z_0)\phi\left(\Gamma(z_0)\right) - \Gamma(z_0)\phi'\left(\Gamma(z_0)\right)}{\phi^2\left(\Gamma(z_0)\right)}$$

$$= \frac{\Gamma'(z_0) - z_0 e\Gamma'(z_0)}{u}$$

Let $T = \Gamma'(z_0)$ and $Q = T - z_0 eT$. To the above equation can be written as $\zeta'(z_0) = Q/u_1$. Here we want to prove that $\rho(\zeta'(z_0) = \rho(Q/u_1) < 1$. First we show that spectral radius of Q is equal to $|v_2|$. It is suffice to show that the spectrum of Q is $\{0, v_2, v_3, \dots, v_N\}$.

We have $1 = \phi(z_0) = (z_0)_1 + (z_0)_2 + \cdots + (z_0)_N = ez_0$, To $ez_0 = 1$ and $Q = T - z_0 eT$, $Q^T e^T = (T - z_0 eT)^T e^T = 0$. Now we can conclude that e^T is an eigenvector of Q^T associated with the eigenvalue 0. There are two possible cases of transpose of matrix , T^T.

Case 1: The matrix $T^T = \Gamma'(z_0)^T$ is diagonizable, that is, T^T is semisimple. For $i = 2,3,\dots,N$, we assume $T^T w^i = v_i w^i$, where w^i is an eigenvector of T^T associated with the eigenvalue v_i. suppose that the set of eigenvector $\{w^1, w^2, \dots, w^N\}$ is linearly independent. To, $v_i z_0^T w^i = z_0^T v_i w^i = z_0^T T^T w^i$, for $i = 2,3,\dots,N$. We have $\Gamma'(z_0)z_0 = Tz_0 = u_1 z_0$. To, $(Tz_0)^T = (uz_0)^T$, and

$$z_o^T T^T = u_1 z_o^T \qquad (2)$$

Hence, $v_i z_0^T w^i = z_0^T T^T w^i = u_1 z_0^T w^i$, $(v_i - u_1) z_0^T w^i = 0$. To, for $i = 2,3,\dots,N$, it is either $v_i = u_1$ or $z_0^T w^i = 0$. But $v_i \neq u_1$ for $i = 2,3,\dots,N$. Therefore $z_0^T w^i = 0$.

Now we have $Q^T w^i = (T - z_0 eT)^T w^i = T^T w^i - 0$. Since we assume $M^T w^i = v_i w^i$, To $Q^T w^i = v_i w^i$. The vector w^i is an eigenvector of Q^T associated with the eigenvalue v_i for $i = 2,3,\dots,N$. Now we prove the set of eigenvectors of Q, $\{e^T, w^2, w^3, \dots, w^N\}$ is linearly independent.

Suppose

$$\alpha_1 e^T + \alpha_2 w^2 + \ldots + \alpha_N w^N = 0, \qquad (3)$$

And $v_i \neq 0$ for $i = 2,3,\dots,p$ and $v_j = 0$ for $j = p+1,\dots,N$. We also have

$$Q^T e^T = 0 e^T,$$

$$Q^T w^i = v_i w^i,\ i = 2,3,\dots,N.$$

Then,

$$Q^T e^T + Q^T w^2 + \cdots + Q^T w^N$$
$$= 0e^T + v_2 w^2 + \cdots + v_p w^p,$$

$$\alpha_1 Q^T e^T + \alpha_2 Q^T w^2 + \ldots + \alpha_N Q^T w^N = \alpha_2 v_2 w^2 + \ldots + \alpha_p v_p w^p. \quad (4)$$

From equation (3),

$$Q^T\left(\alpha_1 e^T + \alpha_2 w^2 + \ldots + \alpha_N w^N\right) = 0,$$

$$\alpha_1 Q^T e^T + \alpha_2 Q^T w^2 + \ldots + \alpha_N Q^T w^N = 0.$$

To the right side of equation (4) became

$$\alpha_2 v_2 w^2 + \cdots + \alpha_p v_p w^p = 0.$$

Since we consider the set $\{w^2, w^3, \ldots, w^N\}$ is linearly independent, $\alpha_2 = \alpha_3 = \cdots = \alpha_p = 0$. From (3),

$$\alpha_1 e^T + \alpha_{p+1} w^{p+1} + \ldots + \alpha_N w^N = 0, \quad (5)$$

$$T^T(\alpha_1 e^T + \alpha_{p+1} w^{p+1} + \cdots + \alpha_N w^N) = 0,$$

$$\alpha_1 T^T e^T + \alpha_{p+1} T^T w^{p+1} + \cdots + \alpha_N T^T w^N = 0.$$

Since $T^T w^i = v_i w^i$ and $v_j = 0$ for $j = p+1, \ldots, N$,

$$\alpha_1 T^T e^T + \alpha_{p+1} v_{p+1} w^{p+1} + \cdots + \alpha_N v_N w^N = 0,$$

$$\alpha_1 T^T e^T = 0.$$

We get $\alpha_1 = 0$ because $T^T e^T > 0$ and T is diagonalizable. From (5),

$$\alpha_{p+1} w^{p+1} + \ldots + \alpha_N w^N = 0. \quad (6)$$

We know that the set $\{w^{p+1}, w^{p+2}, \ldots, w^N\}$ is linearly independent, so $\alpha_{p+1} = \cdots = \alpha_N$. Therefore we get $\alpha_1 = \cdots = \alpha_N = 0$. This means the set $\{e^T, w^2, w^3, \ldots, w^N\}$ is linearly independent. Hence the spectrum of Q is $\{0, v_2, v_3, \ldots, v_N\}$.

Case 2: Consider that T^T is not diagonalizable or defective. We know that a defective matrix has less than N different eigenvalues. Assume that T^T has $q < N$ different eigenvalues $v_1 = \mu_1, v_2, \ldots, v_q$, and these eigenvalues can be written as below.

$$v_1 = \mu_1 > |v_2| \geq |v_3| \geq \ldots \geq |v_q|. \quad (7)$$

So, T^T has the form $T^T = XJX^{-1}$, where $J = diag\{J_1, J_2, \ldots, J_q\}$ is in canonical form. Suppose the square matrices J_i, $i = 1,2, \ldots, q$ be the Jordan blocks with various sizes in the form of

$$J_i = \begin{bmatrix} v_i & 1 & & & \\ & v_i & 1 & & \\ & & v_i & \ddots & \\ & & & \ddots & 1 \\ & & & & v_i \end{bmatrix},$$

where v_i is an eigenvalue of T^T. Let $J_1 = [u_1]$ and X_i is the $i$thcolumn vector of $X, i = 1,2, \ldots, N$. Let l_i is the size of J_i for each Jordan block, where $i = 1,2, \ldots, q$. We have $T^T = XJX^{-1}$, hence

$$T^T X = XJ,$$

$$T^T[X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots] = [X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots] \begin{bmatrix} u_1 & & & & \\ & v_2 & 1 & & \\ & & v_2 & \ddots & \\ & & & v_2 & 1 \\ & & & & \ddots \end{bmatrix}.$$

From the above equation, we get

$$T^T X_2 = v_2 X_2$$
$$T^T X_3 = X_2 + v_2 X_3,$$
$$T^T X_4 = X_2 + v_2 X_4,$$
$$\vdots$$
$$T^T X_{l_2+2} = X_{l_2} + v_2 X_{l_2+1},$$
$$T^T X_{l_2+2} = v_3 X_{l_2+2},$$
$$T^T X_{l_2+3} = X_{l_2+2} + v_3 X_{l_2+2},$$
$$\vdots$$

Just like in Case 1, since $T^T X_2 = v_2 X_2$ and equation (2), $v_2 z_0^T X_2 = z_0^T v_2 X_2 = z_0^T T^T X_2 = u_1 z_0^T X_2, (v_2 - u_1) z_0^T X_2 = 0$. From (7), $v_2 \neq u_1$. To $z_0^T X_2 = 0$. Hence,

$$Q^T = (T - z_0 eT)^T,$$

$$Q^T X_2 = (T - z_0 eT)^T X_2 = T^T X_2 - T^T e^T z_0^T X_2 = T^T X_2 - 0,$$

which means $Q^T X_2 = v_2 X_2$. This implies X_2 an eigenvector of Q^T associated with eigenvalue v_2.

From the equation $T^T X_3 = X_2 + v_2 X_3$, we get $v_2 z_0^T X_3 = z_0^T v_2 X_3 = z_0^T (T^T X_3 - X_2) = z_0^T T^T X_3 - 0$. By equation (2), $v_2 z_0^T X_3 = (u_1 z_0^T) X_3$. Consequently, $(v_2 - \mu_1) z_0^T X_3 = 0$.By (7), since $v_2 \neq u_1, z_0^T X_3 = 0$. Therefore,

$$Q^T = (T - z_0 eT)^T,$$

$$Q^T X_3 = (T - z_0 eT)^T X_3 = T^T X_3 - T^T e^T z_0^T X_3 = T^T X_3 = X_2 + v_2 X_3,$$

Likewise, we get

$$Q^T X_2 = v_2 X_2$$
$$Q^T X_3 = X_2 + v_2 X_3$$
$$Q^T X_4 = X_3 + v_2 X_4$$
$$\vdots$$
$$Q^T X_{l_2+1} = X_{l_2} + v_2 X_{l_2+1}$$
$$Q^T X_{l_2+2} = v_3 X_{l_2+2}$$
$$Q^T X_{l_2+3} = X_{l_2+1} + v_3 X_{l_2+2}$$
$$\vdots$$

same as in Case 1, we show that the set $\{e^T, X_i, i = 2,3, \ldots, N\}$ is linearly independent. Let $Y = [e^T, X_i, i = 2,3, \ldots, N\}]$. Therefore, $Q^T Y = Y diag\{[0], J_2, \ldots, J_q\}$.Now we have the spectrum of $Q, \{0, v_2, v_3, \ldots, v_q\}$ which is similar to the spectrum of Q^T. The spectral radius of Q is $|v_2|$. As a result,

$$\rho(\zeta'(z_0)) = \rho\left(\frac{Q}{\mu_1}\right) = \frac{|v_2|}{u_1} < 1,$$

Since $u > |v_2.|$

Now we determine the convergence rate of Algorithm 1.

Theorem 3. Let A and $\{z_0^{(k)}\}$ be as in Theorem 1. Then the convergence rate of sequence $\{z_0^{(k)}\}$ is Q-linear, that is there exists a vector norm $||\cdot||$ such that

$$\lim_{k\to\infty} \sup \frac{||z^{(k+1)} - z_0||}{||z^{(k)} - z_0||} < 1.$$

Proof. By Proposition 1, there exist an $\epsilon > 0$ and a spectral norm $||\cdot||$ such that

$$\left|\left|\zeta'(z_0)\right|\right|_\epsilon \leq \rho\left(\zeta'(z_0)\right) + \epsilon.$$

By Lemma 1,

$$\left|\left|\zeta'(z_0)\right|\right|_\epsilon \leq \rho\left(\zeta'(z_0)\right) + \epsilon < 1 \qquad (8)$$

Hence, by (6), $z^{(k+1)} = \zeta\left(z^{(k)}\right)$, $k = 1,2,\ldots$, and $z_0 = \zeta(z_0)$. Therefore,

$$z^{(k+1)} - z_0 = \zeta\left(z^{(k)}\right) - \zeta(z_0).$$

Expand $z^{(k)}$ at z_0 using Taylor expansion,

$$\zeta\left(z^{(k)}\right) = \zeta(z_0) + \zeta'(z_0)\left(z^{(k)} - z_0\right) + o\left(\left|\left|z^{(k)} - z_0\right|\right|_\epsilon\right),$$

we have

$$z^{(k+1)} - z_0 = \zeta'(z_0)\left(z^{(k)} - z_0\right) + o\left(\left|\left|z^{(k)} - z_0\right|\right|_\epsilon\right),$$

$$\frac{z^{(k+1)} - z_0}{\left(z^{(k)} - z_0\right)} = \zeta'(z_0),$$

$$\frac{\left|\left|z^{(k+1)} - z_0\right|\right|_\epsilon}{\left|\left|\left(z^{(k)} - z_0\right)\right|\right|_\epsilon} = \left|\left|\zeta'(z_0)\right|\right|_\epsilon,$$

From (8), we get

$$\lim_{k\to\infty} \sup \frac{||z^{(k+1)} - z_0||}{||z^{(k)} - z_0||} < 1.$$

Hence, the convergence rate for Algorithm 1 is linear convergence.

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