ON COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS (PHYSICAL MODEL)

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I. INTRODUCTION

The Navier-Stokes equations are the governing equations for fluid flow. They are a system of nonlinear partial differential equations. There are very few exact solu- tions of the equations, and even for sim- ple geometries, the equations have to be solved numerically. With increasing the Reynolds number, the accurate solution of the equations becomes more difficult, see Ghia et al. (1982), Azzam (2003), Erturk et al. (2005), Boppana & Gajjar (2010) and others. A number of tech- niques and algorithms have been devel- oped to obtain an accurate solution of the equation for high Reynolds numbers. The lid-driven cavity problem is used typically to test new methods and codes. The lid-driven cavity can be introduced as a fluid contained in a square or rectan- gular domain with stationary rigid walls and a moving wall, see Figure (1). The literature review of the two-dimensional and three-dimensional lid-driven cavity reveals that both problems are challeng- ing at a high Reynolds number. Despite the simplicity of the geometry, difficul- ties arise from many complex flow fea- tures, such as the recirculating regions in the corner. This motivates us to ob- tain the numerical solution to the two- dimensional and three-dimensional lid-driven cavity at a high Reynolds number using techniques based on the Chebyshev collocation in two directions to exploit the accuracy it offers. The solution of the two-dimensional Navier-Stokes quations for the lid-driven cavity is presented in many papers. Burggraf (1966) was one of the earliest who stud- ied the problem, using a second order finite difference approximation to solve the equations. Burggraf presented re- sults for Reynolds numbers up to 1000, which in itself was impressive given the computational facilities available at that time.

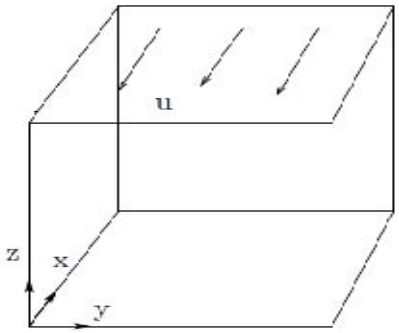


Figure 1: The lid-driven cavity problem

geometry.

A singular behaviour is exhibited in a lot of realistic two dimensional domains which contain sharp corners. The singu- larities are observed when solving the lid driven cavity problem in the vicinity of the two corners of the sliding wall. Mof- fatt (1964) give detailed information of the mathematical and physical proper- ties of what is expected near the corners of the lid driven cavity problem. Usually, when a numerical solution is considered for such flows the singularity is ignored with assumption that the flow is only af- fected in a small region of the singularity, Logue (2005). The Navier-Stokes equations are the governing equations for fluid flow. They are a system of nonlinear partial differ- ential equations. There are very few ex- act solutions of the equations, and even for simple geometries, the equations have to be solved numerically. With increas- ing the Reynolds number, the accurate solution of the equations becomes more difficult, see Ghia et al. (1982), Azzam (2003), Erturk et al. (2005), Boppana & Gajjar (2010) and others. A number of techniques and algorithms have been de- veloped to obtain an accurate solution of the equation for high Reynolds numbers.

II. PROBLEM FORMULATION AND BOUNDARY CONDITIONS

The equations that describe the flow of an incompressible unsteady fluid flow in non-dimensional form are the continuity equation,

$$u_x + v_y = 0, \qquad (1)$$

and the Navier-Stokes equations,

$$u_t + (u.\nabla)u = -\nabla P + \frac{1}{Re}\nabla^2 u \qquad (2)$$

$$v_t + (u.\nabla)v = -\nabla P + \frac{1}{Re}\nabla^2 v, \qquad (3)$$

where (u, v) is the velocity of fluid in the (x, y) directions, P is the fluid pressure and the Reynolds number Re is defined as $\frac{Uw}{\nu}$, where U is the velocity of the lid and w is the cavity width. An alternative formulation of two-dimensional Navier-Stokes equations is to use the vor- ticity (!) and streamfunction ( ) as de- pendent variables. Equations (1), (2) and (3) are expressed in terms of streamfunc- tion (where u = y and v = − x) and vorticity (! = vx − uy), which gives the following formulation,

$$\nabla^2\psi = \omega$$
$$\text{and} \quad \omega_t + \psi_y\omega_x - \psi_x\omega_y = \frac{1}{Re}\nabla^2\omega \quad (4)$$

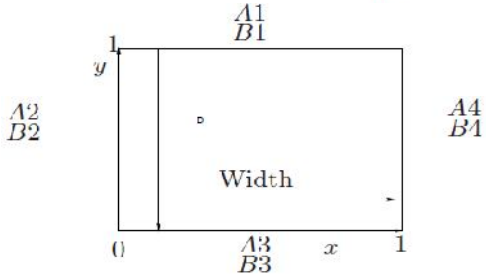


Figure 2: Sketch of the lid-driven cavity boundary condition.

In Figure (2), the various quantities are given as

$$\begin{cases} A1: & \psi = 0, & \text{for} & 0 < x < 1, & y = 1 \\ B1: & \psi_y = 1, & \text{for} & 0 < x < 1, & y = 1 \\ A2: & \psi = 0, & \text{for} & 0 < y < 1, & x = 0 \\ B2: & \psi_y = 0, & \text{for} & 0 < y < 1, & x = 0 \\ A3: & \psi = 0, & \text{for} & 0 < x < 1, & y = 0 \\ B3: & \psi_y = 0, & \text{for} & 0 < x < 1, & y = 0 \\ A4: & \psi = 0, & \text{for} & 0 < y < 1, & x = 1 \\ B4: & \psi_y = 0, & \text{for} & 0 < y < 1, & x = 1. \end{cases} \quad (5)$$

These equations, together with the boundary conditions interpreted on the Figure (2) are sufficient to simulate the lid driven cavity.

At the corners of y = 1, we use a different boundary condition, which is

$$\int_0^1 \left(\frac{\partial^2\psi}{\partial x^2} - \omega\right) dy = 1,$$

while for the boundary conditions of the corners of y = 0, we use

$$\int_0^1 \left[\int_0^y \left(\frac{\partial^2\psi}{\partial x^2} - \omega\right) dy\right] dy = 0.$$

The point of challenging when try simu- lating the lid driven cavity is how to deal with singularities at the top corners. For the corners of y = 1, where the singular- ities take place, the boundary condition- safter using Newton linearisation become

$$\int_0^1 \left(\frac{\partial^2\Psi}{\partial x^2} - \Omega\right) dy = 1 - \int_0^1 \left(\frac{\partial^2\psi}{\partial x^2} - \bar{\omega}\right) dy,$$

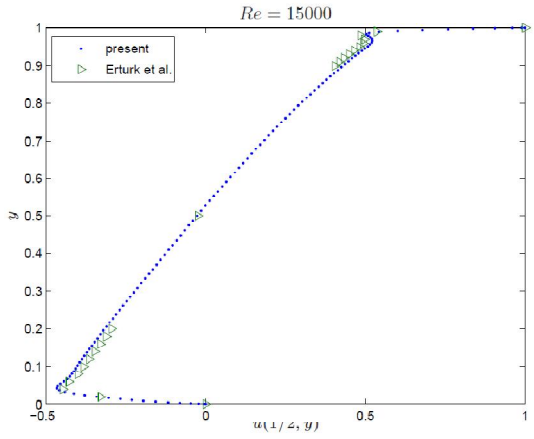
and for the boundary conditions of the corners of y = 0, the boundary condi- tions become

$$\int_0^1 \left[\int_0^y \left(\frac{\partial^2\Psi}{\partial x^2} - \Omega\right) dy\right] dy = -\int_0^1 \left[\int_0^y \left(\frac{\partial^2\bar{\psi}}{\partial x^2} - \bar{\omega}\right) dy\right] dy.$$

One difficulty arises when using the vorticity-streamfunction formulation is the imple- mentation of the no-slip boundary con- dition on the domain. Here, a technique similar to one used by Davies & Carpenter (1997), Azzam (2003), Gajjar & Azzam (2004) and Davies & Carpenter (2001) is employed, where some integral constraints are used instead. These inte-gral relations are derived from the Navier-Stokes equations by integration, whichleads to the compatibility constraints on the vorticity. 3

III. NUMERICAL RESULTS

Numerical results are discussed for the square-driven cavity. As already described, the problem was discretized using the spec-tral collocation method in both direc- tions, x and y, and then solved directly using MATLAB. The upper moving wall boundary condition has been modified to y = 1−e500x(1−x) to avoid the difficulty of a singularity at the corner. Without this modification and solving for the cav- ity problem we could not generate results at high Reynolds number as the code fails to converge. Figures (??,3) show the u-velocity profiles along the vertical lines passing the geometry center of the cavity; x = 1/2. Moreover, the Figure (4) shows the v-velocity profiles along the horizontal lines, passing the geomet- ric center of the cavity; y = 1/2 on a 121 × 121 grid at 5000 ≤ Re ≤ 12500. The high Re profiles of u-velocity show a sharp twist near y = 1, which can be ob- served clearly when Re ≥ 5000 in Figure 3. A similar behavior can be observed in the profiles of v-velocity when x is close to 1. Ghia et al. (1982) have reported the same behavior and mentioned that the kink move towards the upper wall as Re increases, as noted in Figure 3.The u-velocity and v-velocity profiles in Figures (??-4) show a good agreement with the well-known Ghia et al. (1982)study. Furthermore, good agreement is observed with Erturk et al. (2005) re- sults for Re ≤ 12500. For Re ≥ 15000, one can note the deviation between the present results and those of Erturk et al. (2005) as Re increases, see Figure 3. We believe that the present work results are more accurate for two reasons. Firstly, Erturk et al. (2005) generated the results using the finite difference method, while in the present work we use a spec- tral method in both directions.



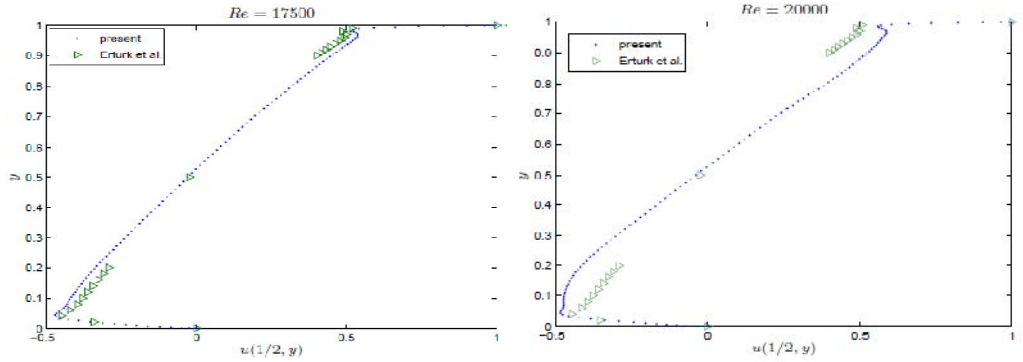


Figure 3: Comparison of u-velocity profiles along a vertical line passing through the ge- ometry centre at various Reynolds numbers using N = M = 120 solution.

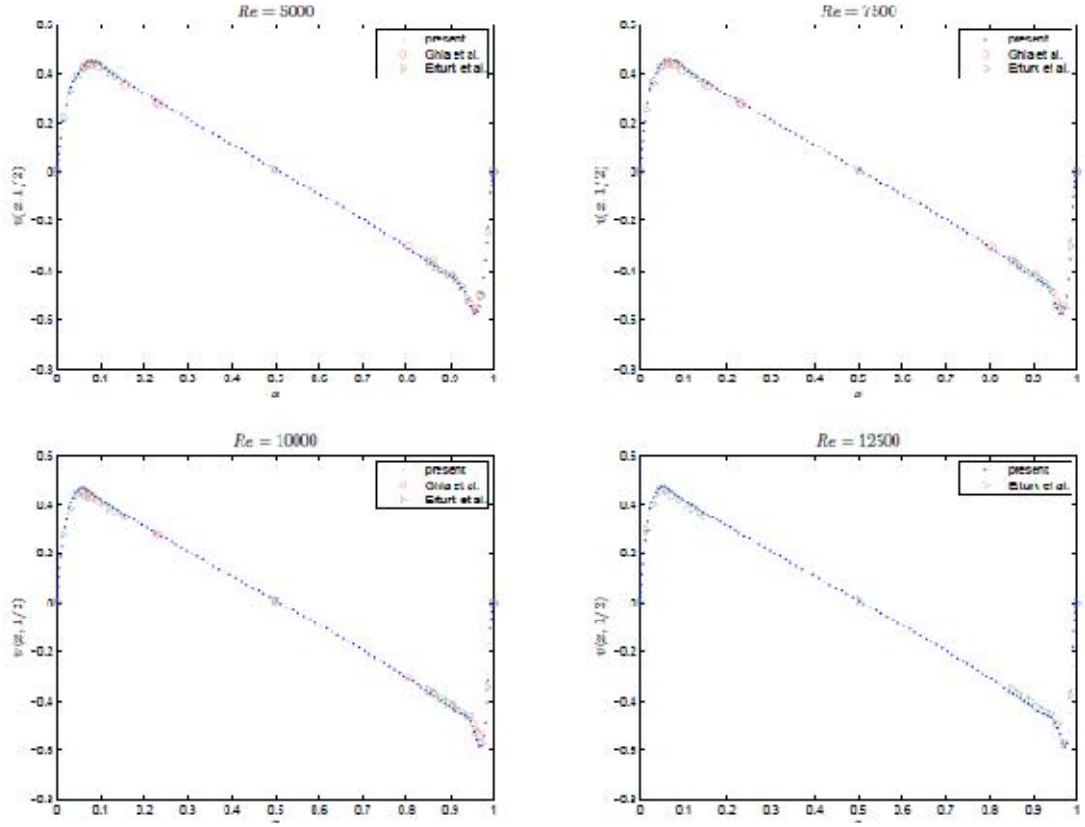


Figure 4: Comparison of v-velocity profiles along a horizontal line passing through the geometry centre at various Reynolds numbers using N = M = 120 solution.

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