FREQUENCY DOMAIN ANALYSIS OF THE QUADRUPLE TANK SYSTEM

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Abstract: - Multivariable systems exhibit complex dynamics because of the interactions between inputs and outputs. The quadruple-tank process has been widely used for control education to present many concepts in multivariable control, such as performance limitations due to multivariable right half-plane zeros or input-output pairing problems. This paper discusses the frequency dependent relative gain array, relative gain array number and Nyquist array plots for the quadruple tank system and addresses diagonal dominance and pole assignment problems.

Key-Words: -Quadruple Tank System, Relative Gain Array, Relative Gain Array Number Plot, Pole Assignment

I. INTRODUCTION

The quadruple tank is a two-input, two output (TITO) system consisting of four interconnected water tanks and two pumps where the two inputs are the voltages to the two water pumps and the two outputs are the water levels of the two lower tanks. The system is shown in Figure 1. Using a three-way valve the output of each pump is split into two and by adjusting the valves between the pumps and the tanks the proportions of water flow to diagonally opposite tanks is changed. In this way the system zero can be moved between the left and right half-planes (RHP) of the complex plane, in other words the system is either minimum or non-minimum phase.

The outline of the paper is as follows. A nonlinear and linearized model for the quadruple tank system is derived in Section II. It is also shown that the valve positions of the system uniquely determine if the system is minimum phase or non-minimum phase. The relative gain array (RGA) and frequency-dependent-RGA is discussed in Section III.

The concepts of relative gain array number (RGA-number) and intersection of RGA-number graphs and controllability indices are presented in Section IV. Diagonal dominance and Nyquist array plots are discussed in Section V. In Section VI pole assignment problem for quadruple tank system is presented. Finally some concluding remarks are given in VII.

II. NONLINEAR AND LINEARIZED MODEL

The mass balances and Bernoulli's law for the process yield as follows, where $\gamma_1 \in [0, 1]$ and $\gamma_2 \in [0, 1]$ are the ratios into which the outputs of the pumps are divided and where $A_i\,[m^2]$ is the cross sectional area of the tank i, $a_i\,[m^2]$ is the cross sectional area of the outlet hole i, and $h_i\,[m]$ is the water level in tank i.

$$\dot{h}_1 = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1 \quad (1)$$

$$\dot{h}_2 = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2 \quad (2)$$

$$\dot{h}_3 = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2 \quad (3)$$

$$\dot{h}_4 = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1 \quad (4)$$

It should be recalled that the voltage applied to pump i is u_i and the corresponding flow is $k_i u_i$. Therefore, the flow to tank 1 is $\gamma_1 k_1 u_1$ and to tank 4 is $(1-\gamma_1)k_1 u_1$. Similarly, the flows for tank 2 and tank 3 are $\gamma_2 k_2 u_2$ and $(1-\gamma_2)k_2 u_2$, respectively. The measured level signals, i.e., the voltages from level measurement devices, are given as $y_1 = k_c h_1$ and $y_2 = k_c h_2$, respectively, where k_c is a constant to represent the conversion factor from water level to voltage level. Let $u_{10}, u_{20}, y_{10}, y_{20}\ h_{10}, h_{20}, h_{30}, h_{40}$ describe an equilibrium for the tanks and let $\Delta u_i = u_i - u_{i0}, \Delta y_i = y_i - y_{i0}$ with $i \in \{1,2\}$, and $\Delta h_j = h_j - h_{j0}$ with $j \in \{1,2,3,4\}$ denote the deviations from the equilibrium.After defining

$$x = (\Delta h_1\ \Delta h_2\ \Delta h_3\ \Delta h_4)^T \quad (5)$$

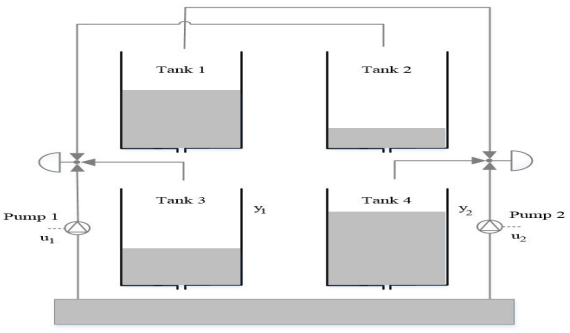


Fig. 1. Quadruple tank system

$$u = (\Delta u_1 \ \Delta u_2)^T \tag{6}$$

$$y = (\Delta y_1 \ \Delta y_2)^T \tag{7}$$

the linearized model is given as $\dot{x} = Ax + Bu$, $y = Cx$ with the following system matrices.

$$A = \begin{bmatrix} \frac{-1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & \frac{-1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & \frac{-1}{T_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{T_4} \end{bmatrix} \tag{8}$$

$$B = \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \tag{9}$$

$$C = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \tag{10}$$

where $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_{i0}}{g}}$. The matrix transfer function from u to y is given by

$$G(s) = \begin{pmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)k_2 c_1}{(1+sT_1)(1+sT_3)} \\ \frac{(1-\gamma_1)k_1 c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{pmatrix} \tag{11}$$

with $c_i = \frac{T_i k_c}{A_i}$, $i = \{1, 2\}$. The McMillan form of $G(s)$ reveals that the poles of the system are located at $s_i = -\frac{1}{T_i}$, $i \in \{1, 2, 3, 4\}$, and the zeros are located at

$$s_{1,2} = \frac{1}{2\gamma_1\gamma_2 T_3 T_4}\left(-\gamma_1\gamma_2(T_3+T_4) \pm \sqrt{\Delta}\right) \tag{12}$$

where

$$\Delta = \gamma_1\gamma_2(4-4\gamma_2)T_3T_4 + \gamma_1(-4T_3T_4 + \gamma_2(T_3+T_4)^2). \tag{13}$$

Recall that if $-\gamma_1\gamma_2(T_3+T_4) \pm \sqrt{\Delta} > 0$ the zeros of $G(s)$ have positive real parts, hence the system is non-minimum phase. It is easy to show that if $-4\gamma_1\gamma_2(\gamma_1+\gamma_2-1)T_3T_4 > 0$ then the system is non-minimum phase. In other words, if $0 < \gamma_1 + \gamma_2 \leq 1$ the system is non-minimum phase, and if $1 < \gamma_1 + \gamma_2 \leq 2$ the system is minimum phase. It should be remembered that $\gamma_1 + \gamma_2$ cannot exceed 2.

III. RELATIVE GAIN ARRAY

In the control theory of multivariable systems, the relative gain array (RGA) is often used for establishing the best input-output pairings. RGA is defined as the Schur product (element-by-element product) of the transfer-function matrix $G(s)$ and the transpose of the inverse of this matrix. Most of the time, the relative gain array is evaluated at zero frequency. For square systems, the relative gain array has the property that all the elements in a row or column sum to unity. To minimize interaction, entries in RGA with relative gains closest to 1 should be paired. Relative gains in the range 0 to 1 indicate moderate interaction, with values of 0.5 being the worst. If the value is greater than 0.5 then the control pair has a greater influence on the system than the other control loops, but if the value is less than 0.5 then the other control loops are influencing the control pair and the influence from the other control loops are greater than the influence from the control pair.It should be noted that entries with negative gains should not be paired and relative gains of greater than 5 mean severe loop interaction. For the quadruple tank system the RGA of $G(0)$ is determined as to be

$$RGA|_{s=0} = \begin{pmatrix} \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} & -\frac{(\gamma_1-1)(\gamma_2-1)}{\gamma_1+\gamma_2-1} \\ -\frac{(\gamma_1-1)(\gamma_2-1)}{\gamma_1+\gamma_2-1} & \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} \end{pmatrix} \tag{14}$$

The relative gain array at zero frequency gives sensible pairings of the inputs and outputs for control in steady state but it neglects the dynamics of the controlled system over the bandwidth of interest. RGA depending on the frequency variable is of the form

$$RGA(s) = \frac{1}{p(s)}\begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix} \tag{15}$$

with

$$p(s) = \gamma_1\gamma_2 T_3T_4 s^2 + \gamma_1\gamma_2(T_3+T_4)s + (\gamma_1+\gamma_2-1) \tag{16}$$

$$\alpha = \gamma_1\gamma_2(T_3T_4 s^2 + (T_3+T_4)s + 1) \tag{17}$$

$$\beta = (\gamma_1-1)(\gamma_2-1). \tag{18}$$

With the help of a new rational polynomial

$$q(s) = \frac{(\gamma_1-1)(\gamma_2-1)}{p(s)} \tag{19}$$

frequency dependent RGA is formulated as

$$RGA(s) = \begin{pmatrix} 1+q(s) & -q(s) \\ -q(s) & 1+q(s) \end{pmatrix}. \tag{20}$$

Here, it is assumed that the best input for the control of output 1 is input 1 and the best input for output 2 is input 2. If a pre-compensator is considered that rearranges the inputs as {2, 1} for outputs {1, 2}, by

applying the appropriate permutation matrix to the system inputs, then frequency dependent the relative gain array becomes

$$RGA_{\{2,1\}}(s) = \begin{pmatrix} -q(s) & 1+q(s) \\ 1+q(s) & -q(s) \end{pmatrix}. \quad (21)$$

IV. RELATIVE GAIN ARRAY NUMBER

The relative gain array number (RGA-number) is a frequency-dependent scalar defined as the sum of the absolute values of the elements of the relative gain array minus the identity matrix;

$$RGA-number = \sum_{i,j=1}^{n} \left| (RGA(jw) - I_n)_{ij} \right| \quad (22)$$

wheren is the size of the array. RGA-number is said to be a better parameter; the smaller the RGA-number is, the lower the interaction is in the system at that frequency.This number is $4|q(jw)|$ and $4|1+q(jw)|$ for the quadruple tank system if the input permutation is {1, 2} and {2, 1}, respectively.If $\gamma_1 = 1$and $\gamma_2 \in [0, 1]$ or $\gamma_2 = 1$and $\gamma_1 \in [0, 1]$ then the RGA-number calculation reveals that the input set {1, 2} is the best at all other frequencies of interest, and the RGA-number is exactly zero for all frequencies.

The parameter values and operating points for the physical system are given in Table 1 and Table 2.

Parameters	Units	Values
A_1, A_3	$[cm^2]$	28
A_2, A_4	$[cm^2]$	32
a_1, a_3	$[cm^2]$	0.071
a_2, a_4	$[cm^2]$	0.057
k_c	$[V/cm]$	0.5
g	$[cm/s^2]$	981

Table 1. Parameter Values for Quadruple Tank

Operating Values	Units	Value
(h_1^0, h_2^0)	$[cm]$	(12.4, 12.7)
(h_3^0, h_4^0)	$[cm]$	(1.8, 1.4)
(v_1^0, v_2^0)	$[V]$	(3.00, 3.00)
(k_1, k_2)	$[cm^2/Vs]$	(3.33, 3.35)
(γ_1, γ_2)	-	(0.70, 0.60)

Table 2. Operating Points for the system

The RGA-number plot is displayed for all permutations of the system inputs for $\gamma_1 = 0.6$ and $\gamma_2 = 0.6$ in Figure 2.

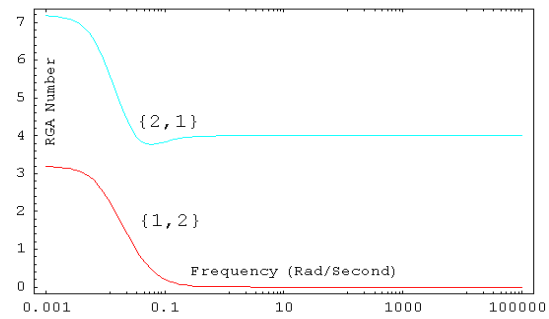


Fig. 2. RGA-Number Plot for $\gamma_1 = 0.6$ and $\gamma_2 = 0.6$

It is obvious that the relative gain array determined for the input set {1, 2} (red) is the best at all other frequencies of interest. For all choices where $\gamma_1 + \gamma_2 = 1$ holds, the RGA-number plot isthe same for all input permutations and the RGA-number is extremely high. This means that all pairings are of same importance. It should also be noted that for this case the system has a zero at the origin. In Figure 3 the RGA-number plot for $\gamma_1 = 0.6$ and $\gamma_2 = 0.2$ is illustrated.

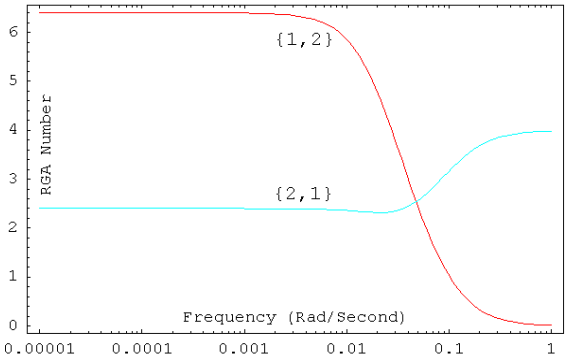


Fig. 3. RGA-Number Plot for $\gamma_1 = 0.6$ and $\gamma_2 = 0.2$

Here, it can be observed that the input-output pairing should be changed for relatively high frequency dynamics. As can be seen from Figure 3 the RGA-number plots may intersect at some points, which results in some frequency band where the input-output pairings must be switched. These critical frequencies are determined by solving the equation

$$4|q(jw)| = 4|1+q(jw)| \quad (23)$$

which results in $\mathrm{Re}[q(jw)] = -0.5$ and $\mathrm{Im}[q(jw)]$ is arbitrary. For the case represented in Figure 3 this critical frequency is $w_{cr} = 0.0479947586$ rad/s. In general, the cricitical frequency is the positive real root w to the polynomial $c + b\,w^2 + a\,w^4$ where

$$a = \gamma_1^2 \gamma_2^2 T_3^2 T_4^2 \quad (24)$$

$$b = \gamma_1^2 \gamma_2^2 (T_3^2 + T_4^2) \quad (25)$$

$$c = (\gamma_1 + \gamma_2 - 1)(2\gamma_1\gamma_2 - \gamma_1 - \gamma_2 + 1). \quad (26)$$

If $c = 0$ then the only critical frequency that both RGA-number plots intersect is $w = 0$ which means again that $\gamma_1 + \gamma_2 = 1$.As a, b and w are all real and positive numbers c has to be negative, which means that $\gamma_1 + \gamma_2 < 1$. As a conclusion we can state that the two RGA-number plots intersect if and only if $\gamma_1 + \gamma_2 \le 1$ holds.

V. COLUMN DIAGONAL DOMINANCE

A rational $p \times p$ matrix $G(s)$ is columndiagonal dominant on theNyquistD-contourif

$$|g_{ii}(jw)| > \sum_{i=1}^{p} |g_{ij}(jw)| \quad \forall s \text{ on } D \tag{27}$$

for $j = 1, 2, ..., p$ but $i \neq j$.

Thecolumndiagonaldominance of a system can be determinedgraphicallybysuperimposingtherowdominanceGershgorincircles on thediagonalterms of theso-calledNyquistarray[3], that is, foreachvalue of frequency w_k a circle of radius

$$r_i^c = \sum_{i=1}^{p} |g_{ij}(jw_k)| \tag{28}$$

for $j = 1, 2, ..., p$ but $i \neq j$, are drawn with centers on the diagonal element $g_{ij}(s)$ evaluated at $s = jw_k$. A system is column diagonal dominant over the bandwidth of interest, if none of the bands generated by the Gershgorin circles superimposed on the diagonal terms of the Nyquist array enclose the origin of the associated diagonal element plot[4]. In our case, choosing $\gamma_1 = 0.3$ and $\gamma_2 = 0.7$ results in a Nyquist array plot given in Figure4.

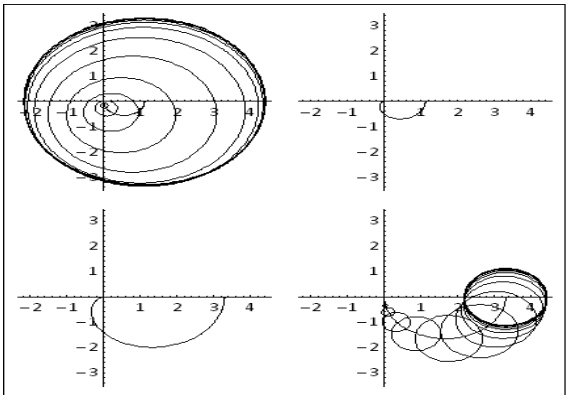


Fig. 4. Nyquist Array Plot for $\gamma_1 = 0.3$ and $\gamma_2 = 0.7$

On the other hand if $\gamma_1 = 0.7$ and $\gamma_2 = 0.3$ then the Nyquist array plot is as given in Figure 5.

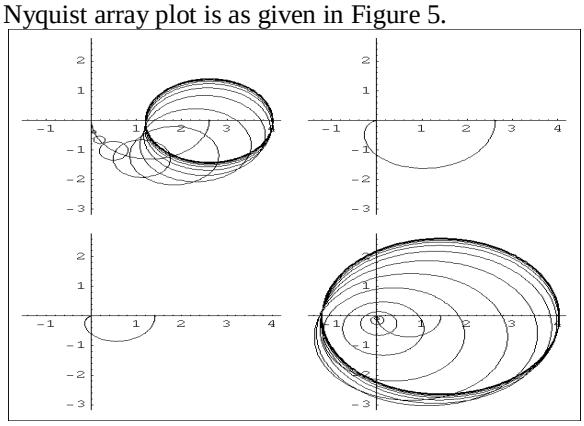


Fig. 5. Nyquist Array Plot for $\gamma_1 = 0.7$ and $\gamma_2 = 0.3$

As can be seen in Figure4 the Gershgorin column dominance circles for the low-frequency part of element {1,1} of the Nyquist array, which starts at 1.12 on the real axis for w=0 enclose the origin, therefore the system is not column diagonal dominant at low frequencies, but those for element {2,2} of the Nyquist array plot, which starts at 3.31 on the real axis for w = 0, do not enclose the origin at low frequencies. However, the decreasing size of the Gershgorin circles of element {1,1} means column diagonal dominance at the higher frequencies. The simulation plot in Figure 6 of the open-loop system to unit step input also confirms that.

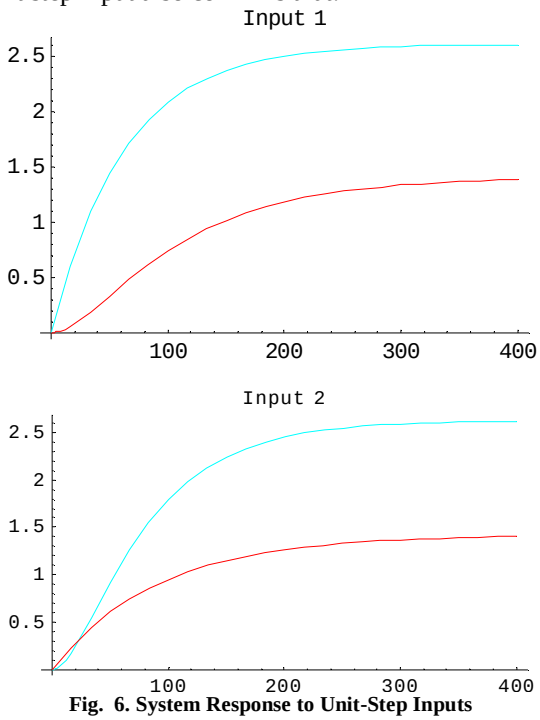


Fig. 6. System Response to Unit-Step Inputs

Choosing $\gamma_1 = 0.6$ and $\gamma_2 = 0.7$ results in a Nyquist array plot given in Figure7.

Futhermore it can be shown that for $\gamma_1 + \gamma_2 < 1$ the so called Perron-Frobenius eigenvalue is greater than 2, therefore the system cannot be made diagonally dominant at low frequencies.As an example let us consider the case with $\gamma_1 = 0.6$ and $\gamma_2 = 0.2$, where the Perron-Frobenius eigenvalue plot is given in Figure 8. Furthermore, as the controllability test reveals, the system is state controllable, however one should note that the controllability indices are {2, 2}, which means that the quadruple tank system is controllable if and only if both input channels are active and if $\gamma_1 = \gamma_2 = 1$ then these indices are {1, 1}, in other words the controllable subspace is two dimensional.

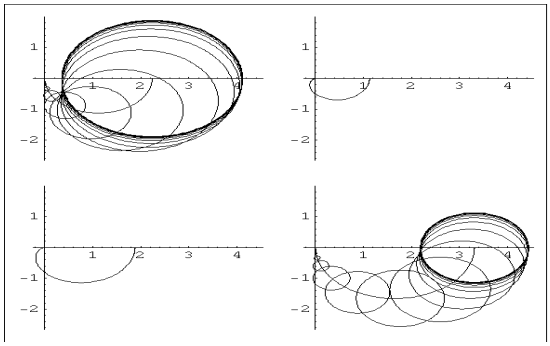


Fig. 7. Nyquist Array Plot for $\gamma_1 = 0.6$ and $\gamma_2 = 0.7$

On the other hand if only $\gamma_1 = 1$ or only $\gamma_2 = 1$ holds, then the controllability indices are {1, 2} and {2, 1}, respectively, which reveals that the controllable subspace is three dimensional.

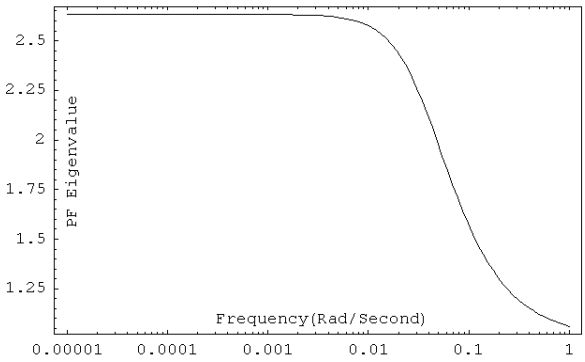


Fig. 8Perron-Frobenius Eigenvalue Plot for $\gamma_1 = 0.6$, $\gamma_2 = 0.2$

In Table 3 the controllability incides are presented in detail.

γ_1	γ_2	Controllability Index
0	1	{1, 1}
1	0	{1, 1}
1	1	{1, 1}
1	$0 < \gamma_2 < 1$	{1, 2}
$0 < \gamma_1 < 1$	1	{2, 1}
$0 < \gamma_1 < 1$	0	{2, 2}
$0 < \gamma_1 < 1$	$0 < \gamma_2 < 1$	{2, 2}
0	$0 < \gamma_2 < 1$	{2, 2}
0	0	{2, 2}

Table 3. Controllability incides

VI. POLE ASSIGNMENT FOR QUADRUPLE TANK

The pole assignment problem is the problem of finding a feedback matrix K for the system $\dot{x} = Ax + Bu$, such that $u = -Kx$ and the matrix $A - BK$ has a desired set of eigenvalues.There are some methods for the pole assignment problem such as the recursive single-input algorithm and a modified version of the recursive multi-input algorithm , the RQ implementation of the recursive single-input algorithm , the explicit QR algorithm ,and the implicit single-input RQ algorithm .These methodsfollow the following procedure:

The pair (A, B) is first transformed to the controller-Hessenbergpair $(H, \tilde{B})$. A feedback matrix F is computed such that the matrix $H - \tilde{B}F$ has the desired set of eigenvalues.A feedback matrix K for the original problem is calculated from the matrix F of the Hessenberg problem in previous step: $K = FT$, where T is the orthogonal transforming matrix that transforms the pair (A, B) to the pair $(H, \tilde{B})$.

After transforming the given state-space system to the block controller-Hessenbergform $(H, \tilde{B})$, a block upper-triangular matrix L is constructed with the help of the recursive algorithm and a block upper-bidiagonal matrix Λ, whose diagonal blocks Λ_{21}, Λ_{32}, … , $\Lambda_{k,k-1}$ contain the eigenvalues to be assigned, such that

$$LH - \Lambda L = L\tilde{B}F = \begin{pmatrix} L_{11}RF \\ 0 \end{pmatrix} \tag{29}$$

where $\tilde{B} = \begin{pmatrix} R \\ 0 \end{pmatrix}$ and R has full rank.After that set $L_k = \begin{pmatrix} 0 \ldots 0 & I \end{pmatrix}$ and the blocks of L are computed recursively as $\tilde{L}_i = L_{i+1}H - \Lambda_{ii}L_{i+1}$. This step is followed by the RQ decomposition of $\tilde{L}_i$ and the feedback matrix F is determined by solving the linear system $L_{11}RF = L_1H - \Lambda_{11}L_1$ .Although the recursive algorithm is the most efficient of all the algorithms, it is not necessarily numerically stable. In the single-input case, however, the algorithm has been proven to be numerically reliable in the sense that the ill-conditioning of the matrix L is an indicator of the breakdown of the algorithm. Both the single-input and multi-input algorithms have worked well in most test cases. The RQ version of the single-input algorithm is numerically stable . For a fixed γ_1 and γ_2 the system state space model is presented as follows.

$$A = \begin{pmatrix} -0.24038 & 0 & 0.1310 & 0 \\ 0 & -0.2852 & 0 & 0.0873 \\ 0 & 0 & -0.1310 & 0 \\ 0 & 0 & 0 & -0.0873 \end{pmatrix} \tag{30}$$

$$B = \begin{pmatrix} 0.0245 & 0 \\ 0 & 0.02115 \\ 0 & 0.0141 \\ 0.0105 & 0 \end{pmatrix} \tag{31}$$

$$C = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{pmatrix} \tag{32}$$

The pole assignment problem for the desired closed-loop poles {-6,-5,-1+0.2j,-1-0.2j} is solved and the following state feedback matrices are found for recursive and explicit QR algorithms, respectively.

$$K_1 = \begin{pmatrix} -1203.87 & -988.453 & 1447.41 & 3431.87 \\ -77.4553 & -1311.37 & 2372.48 & 165.711 \end{pmatrix} \tag{34}$$

$$K_2 = \begin{pmatrix} -190.952 & -96.5654 & 143.031 & 605.508 \\ 7317.01 & -2620.71 & 4681.18 & 795.368 \end{pmatrix} \tag{35}$$

CONCLUSIONS

The Quadruple-Tank Process is used today in several courses in the control education. This study investigated the frequency dependent RGA and diagonal dominance problem for this system. As a

future work we will solve the problem of all stabilizing controllers for the given system.

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