BEST LINEAR PREDICTION FOR FUNCTIONAL AUTOREGRESSIVE PROCESSES GIVEN BY PROJECTION IN LINEARLY CLOSED SUBSPACES

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Abstract - We consider the Best Linear Predictor (BLP) of Functional Autoregressive Processes built with orthogonal projection on linearly closed subspaces introduced by R.Fortet.
We show almost sure convergence and uniform convergence of the predictors. We then provide exponential bounds for the predictors extending and improving previous results existing in the prediction of C[a,b] space and auto-reproducing kernel space valued processes.
We illustrate the finite sample performance of the predictors by a simulation study and through real examples from climatology comparing with others prediction methods existing in the literature.
This simulation study enlighten on the link between the convergence rates of predictors BLP and the presence of the first eigenvalues of the covariance operator and sample size.

Keywords - Functional Autoregressive Processes, enclosed subspace, measurable linear transformations, orthogonal projection, El Nino series.

I. INTRODUCTION

The purpose of this note is the determination of the best linear approximation λ_0 (X) of a random variable Y, given X , where X and Y values in a real separable Hilbert space (H, ‖.‖) . It is therefore minimized $E\|Y - l(X)\|^2$, where l is a linear operator on H.
The problem can then treat as follows : we place in Hilbert space random variables squared standard Hilbert intégrableet seeking to determine the orthogonal projection of Y on the souse enclosed space G(X) generated by X. The solution is then a measurable linear transform (MLT) .This gives an explicit form of λ_0 (X) , in direct form in the Gaussian case and in the autoregressive case, after we prove the convergence almost sure of this predictor . The part of simulation confirms our findings.

II. MATERIAL AND METHODS

- H a Hilbert real and separable space equipped with the standard norm ‖.‖.
- $(X_1,...,X_n)$ an ARH(1) sequence of random values in H such that $X_n = \rho X_{n-1} + \varepsilon_n$, $n \in Z$ ….(1) Where ε_n is an H-white noise and ρ is a compact operator and symmetric on H such as $\rho^{j_0} \in L^1$ for $j_0 \geq 1$.
- For Hilbertiens random variables, the operators of covariance and cross-covariance on H are defined by:

$$C(x) = C_{X_0}(x) = E(\langle X_0, x\rangle X_0) \quad \text{and}$$
$$D(x) = C_{(X_0,X_1)}(x) = E(\langle X_0, x\rangle X_1)$$

- For an ARH(1) we have D=ρC.
- The corresponding empirical covariance operator of C and D are defined respectively by C_n and D_n:

$$C_n(x) = \frac{1}{n}\sum_{i=1}^{n}\langle X_i, x\rangle X_i \,, x \in H$$

$$D_n(x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle X_i, x\rangle X_{i+1} \,, x \in H$$

$(\lambda_j, v_j)_{j\geq 1}$ be the eigenelements of C and the eigenelements of C_n.
(Ω,A,P) a probabilised space
$L_H^2 = L_H^2(\Omega, A, P) = \{Z: \Omega \to H; E\|Z\|^2 < \infty\}$,
equipped with scalar product
$(Z_1, Z_2) = E\langle Z_1, Z_2\rangle; Z_1, Z_2 \in L_H^2$
we will assume that L^2_H is separable.
$G(X) = \overline{sp}\{l(X); l \in \mathcal{L}\}$ is The enclosed subspace generated by a random variable X, where $X \in L^2_H$; E(X)=0 and $\mathcal{L}=\mathcal{L}(H,H)$ is the set of linear applications bounded on H.

$\varepsilon_n X_n = X_n - E(X_n/X_n) = \varepsilon_0 X_n$ where
$\varepsilon_0 X_n = \frac{1}{n}\sum_{i=1}^{n} \langle X_i, v_i\rangle D_n v_i$

Subsequently we note $C=C_{X0}$, $D= C_{X_0,X_1}$
Let $X_1,...,X_n$ be a sample of ARH(1). The corresponding empirical covariance operator of C and D are defined respectively by C_n and D_n:

$$C_n(x) = \frac{1}{n}\sum_{i=1}^{n}\langle X_i, x\rangle X_i, \qquad \forall x \in H.$$

1. Under confined space and measurable linear transformations

we put (Ω,A,P) a probabilised space and H an Hilbert real space separable equipped with its norm‖.‖, of his scalar product <.,.> and his borellien tribe B_H, on pose $L^2_H= L^2_H (\Omega,A,P)=\{Z:\Omega\rightarrow H;E^2<\infty\}$ equipped with scalar product

$(Z_1,Z_2)=E<Z_1,Z_2>$; $Z_1,Z_2\in L^2_H$

we will assume that L^2_H is separable.

Definition: The enclosed subspace generated by a random variable $Y:\Omega\rightarrow H$ is defined by

$G(Y)=sp\{l(Y);l\in\mathcal{L}\}$

where $Y\in L^2_H$;E(Y)=0 and $\mathcal{L}=\mathcal{L}(H,H)$ is the set of linear applications bounded on H.

We consider the spectral development of the covariance operator of Y

$C(Y)=\sum \alpha_i v_i\otimes v_i$

where $\alpha_i>0$; $\sum\alpha_i<\infty$ and $(v_i)_i$ is an orthonormal system in H.

Proposition Let $(X_n,n\in\mathbb{Z})$ be an ARH(1).Then

$\Pi^{X_n}(X_{n+1}) = E(X_{n+1}/X_n)=\lambda_0(X_n)$

where $\lambda_0(X_n) =\sum \frac{1}{\alpha_i}<X_n,v_i> D_n(v_i)$

III. RESULTS AND DISCUSSION

The convergence demonstrated is linked to the predictor $\widetilde{\rho(Xn)}$ obtained by orthogonal projection on enclosed subspace generated by X_n (in the Gaussien and ARH(1) cases) that Bosq-Mourid have found in 2013:

1. Eigenvectors of the covariance operator C are known

Let (α_j,v_j) pour j≥1 be the eigenelements of C and $(\alpha_{jn})_{(j\geq1)}$ the eigenvalues of C_n. It is assumed that v_j are known and the α_j are estimated from α_{jn}. We may define an estimator of ρ by setting:

$$\widetilde{\rho(X_n)} = \frac{1}{n}\sum_{i=1}^{kn}\sum_{j=1}^{n}\frac{1}{\alpha_{in}}\langle X_n,v_i\rangle\langle X_j,v_i\rangle X_{j+1}$$

where $k_n\nearrow\infty$.

We now make the following assumptions

H_1 :

1) $E\|X_0\|^4<\infty$

2) $p(\alpha_{n1}>\ldots>\alpha_{kn}>0)=1$

3) $\alpha_1>\ldots>\alpha_n>0$

We have the following result:

Theorem1 Let $X=(X_n;n\in\mathbb{Z})$ be a standard ARH(1) process, and there exists $N_n>0$ such that $\|X_n\|=N_n$ under H_1 and if

$$\frac{N_n k_n^{\,2}}{\alpha_{kn}} = 0\left(\frac{n}{log\,(n)}\right)^{\frac{5}{2}}\ when\ n\rightarrow+\infty$$

where $(k_n)_n$ the sequence of naturals numbers $(k_n)\nearrow\infty$.

We have

$\|\widetilde{\rho(Xn)}-\rho(X_n)\|\rightarrow0$ when $n\rightarrow+\infty$ a.s .

2. The eigenvectors of the covariance operator C are unknown

Let $X=(X_n;n\in\mathbb{Z})$ be a standard ARH(1) process, such that $\|X_n\|<M$ and (α_{jn},v_{jn})be the eigenelements of C_n . we define

$$\widetilde{\rho(X_n)} = \frac{1}{n}\sum_{i=1}^{kn}\sum_{j=1}^{n}\frac{1}{\alpha_{in}}\langle X_n,v_{in}\rangle\langle X_j,v_{in}\rangle X_{j+1}$$

where $(k_n)\nearrow\infty$.

Theorem2

We make the following assumptions

H_2 :

$$\alpha_{kn}^{-1}\sum_{j=1}^{kn}a_j = \bigcirc\left(\frac{n^{\frac{1}{4}}}{(log\,(n))^{\beta}}\right) when\ n\rightarrow+\infty$$

if the assumptions H_1 and H_2 are hold, then

$$\sup_{\|Xn\|<M}\left\|\widetilde{\rho(Xn)}-\rho(Xn)\right\|\rightarrow0\ when\ n\rightarrow+\infty\ a.s$$

IV. EXAMPLE AND SIMULATION

Example 1

Conditions of Theorem1 are satisfied if we take eigenvalues of the covariance operator

$\alpha_j=a.j^{-\delta}$, j≥1

where , a>0, δ>1, $k_n=[n^{v}]$ with $v<\frac{1}{8\delta}$, δ>1 and $N_n\leq M$, M>0. Indeed we deduce that

$$\frac{k_n^2(logn)^{\frac{5}{2}}}{k_n^{-2\delta}n^{\frac{1}{2}}}\leq\frac{n^{2v(\delta+1)-1/2}\,(log\,(n))^{\frac{5}{2}}}{a}$$

and the last bound tends to zero if $2v(\delta+1)-(1/2)<0$ which is true under our conditions.

Example 2

Conditions of Theorem 2 are satisfied if we take eigenvalues of the covariance operator

$\alpha_j=a.j^{-\delta}$, j≥1

where , a>0, δ>1, $k_n=\log n$. Indeed we show that for large n

$$\frac{k_n(logn)^{\beta}}{\alpha_{kn}^3n^{\frac{1}{2}}}=\frac{(logn)^{\beta+1}}{a^3(logn)^{-3\delta}n^{\frac{1}{2}}}=\frac{(logn)^{\beta+1-3\delta}}{a^3n^{\frac{1}{2}}}\rightarrow0\ where\ n\rightarrow+\infty$$

Simulations d'un ARH(1) :

We illustrate the behavior of the predictor by numerical simulations in some cases. To simulate a trajectories of ARH(1) process given by (1) where $H=L^2[0,1]$ we use R-package far developed by J. Damon and S. Guillas damons [??] which uses the increments of Brownian motion to generate a strong white noise (example 1 and 2 below). We take the following spectral decomposition for ρ

and $e_k(t)=\sqrt{2}\sin((k-1/2)\pi t)$ and $\sigma^2_k=\frac{1}{\left(k-\frac{1}{2}\right)^2\pi^2}$. Here $(\sigma^2_k\,,\,e_k(t))$ are eigenvalues and eigenfunctions of the covariance operator Cε of Brownian motion operateur à noyau $K(t)=\frac{1}{10^2}\cos(t)$

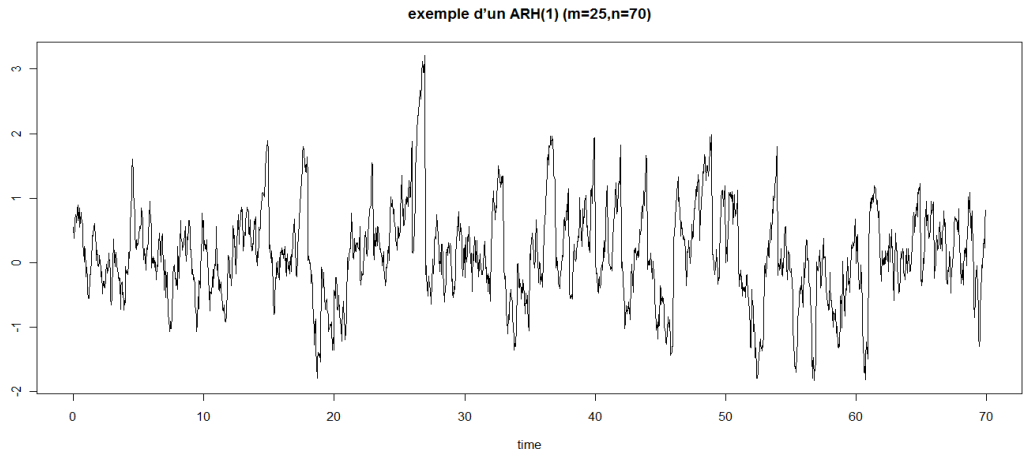

Figure 1

Fig. 2: Prediction of a ARH(1)
Eigenvectors of the covariance operator C are known

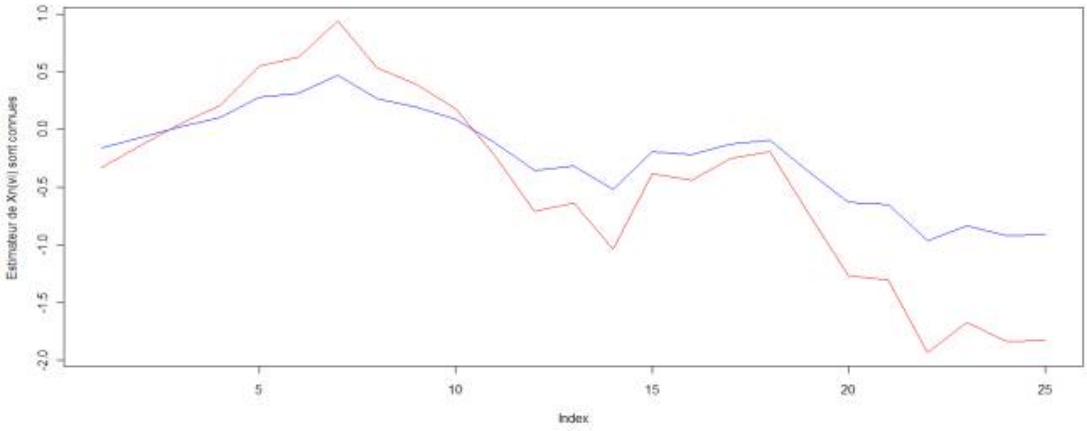

Figure 2

n=100, m=25, k_n=2
inred:$\widehat{\rho(Xn)}$, inblue:X_{n+1},
EQ=0.445732

Fig. 3: Prediction of a ARH(1)
Eigenvectors of the covariance operator C are unknown

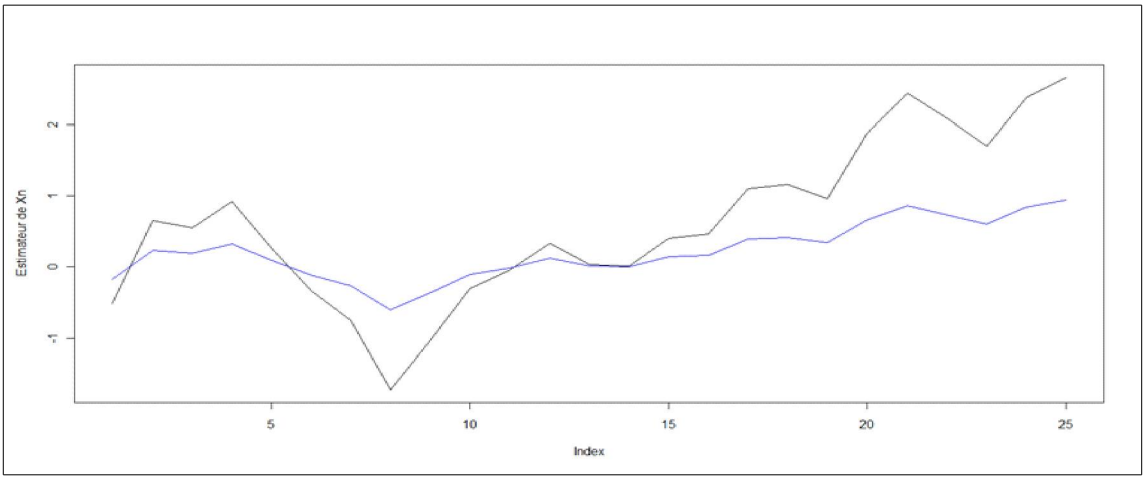

Figure 3

n=70, m=25, k_n=2
in black:$\rho(\widetilde{Xn})$, in blue:X_{n+1},
EQ=1.387241

Fig. 4 table1 : Example of NINO 3 2006

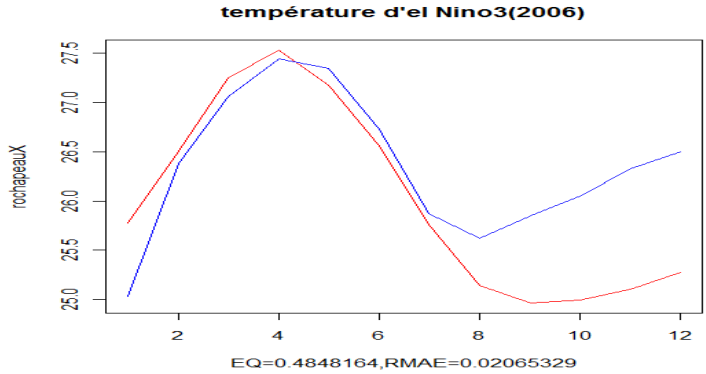

Figure 4

température of nino3 2006
EQ = 0.4646164, RMAE=0.0206532

Méthodes de prévision	RMAE
Climatologie	2.5%
SARIMA	3.7%
Kernel	2.3%
Functional kernel	2.2%
Smooth FAR(1)	2.3%
Smooth FAR(1) with q=p=12	2.4%
Local FAR(1)	2.2%
BLUP discret	1.40%
Blup continu	2.25%
Predicteur sieves	2.46%
Predicteur via les sous espaces clos	2.06%

Table1: RMAE errors for prediction of ElNino index during 2006

Example of Atmospheric concentration of CO2

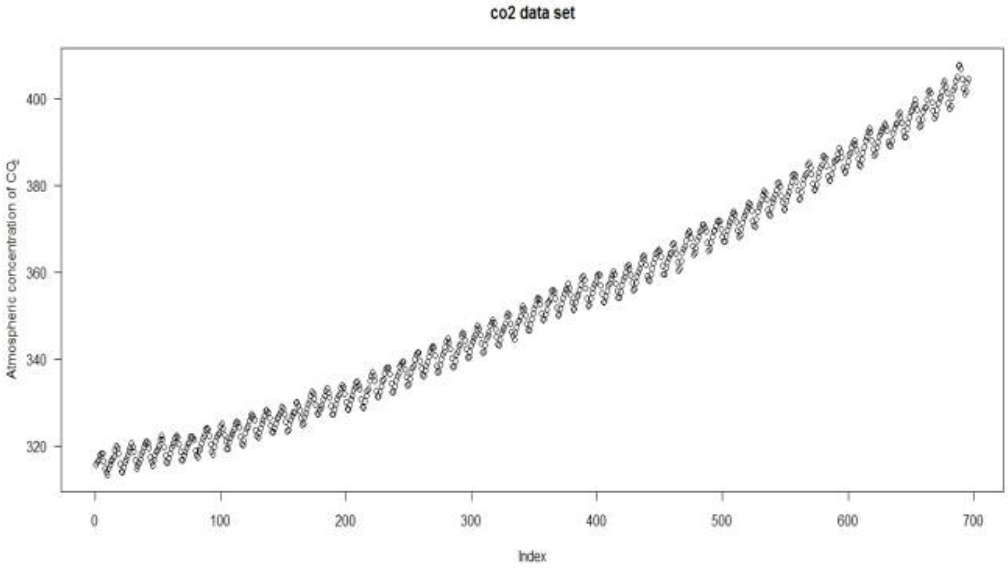

Fig. 5: Atmospheric concentration of CO2

Figure 5
Fig. 6 Fig. 7 : Atmospheric concentration of CO2 years 1992, 2015 respectively

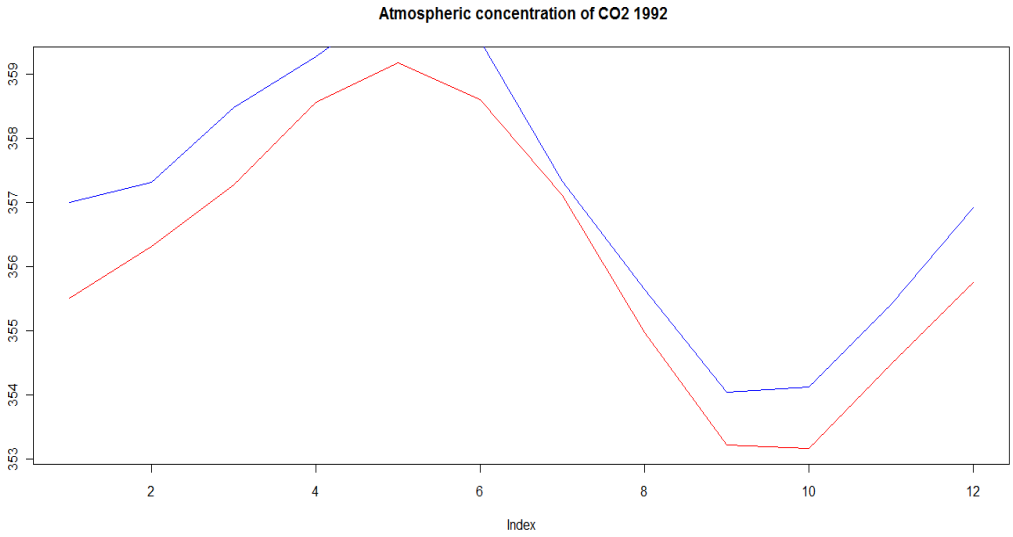

Figure 6

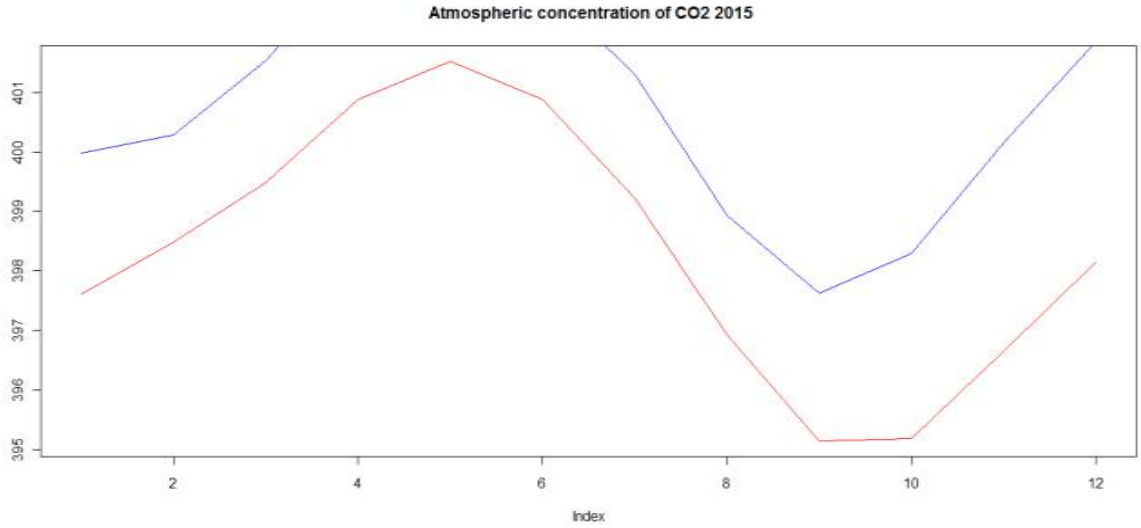

Figure 7

Remark that the predictor obtained by the orthogonal projection on the enclosed subspace has an RMAE error comparable to other methods except the discreet BLUP

CONCLUSION

Our work has improved the results of convergence achieved by other authors.
The simulation part confirms our results.

REFERENCES

[1] Bosq D. Linear Processes in Function spaces. Theory and application. Lecture Notes in Statistics 149 Springer 2000
[2] Bosq, D.(2007). General linear processes in Hilbert spaces and prediction.J. Stat. Planning and Inference,137879-894.
[3] Bosq, D., Blanke D.(2007).Inference and prediction in large dimensions.Wiley series in probability and statistics, John Wiley and Sons, Dunod.
[4] Mourid T. 1995. Thèse de Doctorat d'Etat eb Sciences, Université paris6
[5] Mokhtari, F., Mourid, T.(2002) Prediction of autoregressive processes via the reproducing kernel spaces.C. R. Acad. Sci. Paris, Ser.,334 65-70.
[6] Mourid T.(1995). Contribution à la statistique des processus autoregressifs a temps continu. PHD. Thesis, Univ. Paris VI.
[7] Mourid, T. (2002). Estimation and prediction of functional autoregressive processes.Statistics,36 125-138.
[8] Mourid, T.(2004). The hilbertian autoregressive process with random oparator (in french).Annales de l'ISUP,48 79-86.
[9] Mourid T., Bensmain, N.(2006). Sieves estimator of the operator of a functional autoregressive process. Stat. and Probab. Letters,76 93-108.
[10] Bosq D, Mourid T Une note sur les sous-espaces clos Gaussiens Ann. ISUP, 2013, 57, 3, 103-115.