NUMERICAL SOLUTION OF AN INTEGRAL EQUATION OF CHANDRASEKHAR TYPE IN THE THEORY OF RADIATIVE TRANSFER

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Abstract - We are concerned here with a nonlinear quadratic integral equation (QIE) of Fredholm type. The existence of solutions will be proved. Convergence analysis of Adomian decomposition method (ADM) applied to these type of equations is discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of Adomian's series solution.
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I. INTRODUCTION

Radiative transfer in a plane parallel atmosphere has been a major scientific and mathematical subject for many years. Chandrasekhar's work in radiative transfer theory began in 1944 and culminated with the publication of his influential treatise Radiative Transfer in 1950. In [18] his major contributions to radiative transfer were recounted and evaluated. Many problems of radiative transfer have been reduced by S. Chandrasekhar, ([13], Chapters 4 and 5; 3; 4), to the problem of solving the integral equation

$$\frac{1}{x(t)} = 1 - t\int_0^1 \frac{\phi(s)x(s)}{t+s}\,ds$$

where $\phi(s)$ is known and $x(t)$ is to be found.
Chandrasekhar's historical work in 1960 formulated the radiative transfer process and derived the simultaneous integral equation for the scattering function and transmitted functions. Chandrasekhar's Simultaneous Integral Equation was introduced, and the iteration scheme for the scattering and transmitted functions were given in [20]. Quadratic integral equations (QIEs) are often applicable in the theory of radiative transfer, kinetic theory of gases, theory of neutron transport and traffic theory [1]-[9] and [22]. In this paper, ADM is used to solve QIEs. This method has many advantages, it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization [1]-[6], [10]-[12], [16] and [17]. Here, we deal with the nonlinear QIE

$$x(t) = a(t) + x(t)\int_0^1 \frac{t}{t+s} f\big(s,x(s)\big)ds, \quad t \in [0,1] = I. \quad (1) \qquad (1)$$

The problem in [1]

$$x(t) = 1 + \lambda x(t)\int_0^1 \frac{t}{t+s}\varphi(s)x(s)ds,$$

is a special case of our problem.
The contribution of the work reported in this paper can be summarized in the following four points:

- Introducing the sufficient condition that guarantees the existence of unique solution to our problem.
- Based on the above point using Adomian polynomials formula suggested in [16] and [17], convergence of ADM is discussed (see Theorem 2).
- The maximum absolute truncated error of the Adomian's series solution is estimated (see Theorem 3).
- Preparation of an algorithm using MATHEMATICA package to solve the given numerical examples.

II. EXISTENCE THEOREM

Consider the assumptions:
i. a: $I \to \mathbb{R}$, are continuous.
ii. f: $I \times \mathbb{R} \to \mathbb{R}$, is continuous and $M = \sup_{(t,x)\in I\times\mathbb{R}} |f(t,x)|$, and there exists a constant l satisfies

$$|f(t,x) - f(t,y)| \le L\,|x-y|,$$

iii. for all $t \in I$ and $x, y \in \mathbb{R}$. item[(iii)] $M + L\,r < 1$.
Let $C = C(I)$ be the class of all real functions defined and continuous on I with the norm

$$|| x || = \max\{ |x(t)| : t \in I \}.$$

Define the operator T by

$$Tx(t) = a(t) + x(t) \int_0^1 \frac{t}{t+s} f(s, x(s))\, ds, \; t \in I,$$

Theorem 2.1. Let the assumptions (i)-(iii) be satisfied, then the quadratic integral equations of Chandrasekar type (1) has unique solution in C.

Proof

Define

$$U = \{x(t) | x \in C: ||x||_C \leq r\}.$$

For $x, y \in U$, we deduce that

$$\begin{aligned} |Ty(t) - Tx(t)| &\leq |y(t) \int_0^1 \frac{t}{t+s} f(s, y(s))\, ds \\ &\quad - x(t) \int_0^1 \frac{t}{t+s} f(s, x(s))\, ds| \\ &\leq |x(t)| \int_0^1 \frac{t}{t+s} |f(s, y(s)) - f(s, x(s))|\, ds \\ &\quad + |y(t) - x(t)| \int_0^1 \frac{t}{t+s} |f(s, y(s))|\, ds \\ &\leq (M + L\, r)\, ||y - x|| \leq \alpha\, ||y - x||, \; 0 < \alpha = (M + rL) < 1. \end{aligned}$$

Hence, we conclude that the quadratic integral equation (1) has a unique solution by the Contraction mapping principle. This ends the proof.

III. THE PROBLEM WITH THE SOLUTION ALGORITHM

Adomian polynomial algorithm has been extensively used to solve linear and nonlinear problems arising in many interesting applications (see, e.g., [1]-[6] and [10]-[21]). The algorithm (a decomposition method) assumes a series solution for the unknown quantity. It has been shown [11] that the series converges fast, and with only few terms, this series approximates the exact solution with a fairly reasonable error. In [15], E. Deeba and Sh. Xie adapt the algorithm and a modification version of the algorithm due to Wazwaz [21] to the solution of the nonlinear Volterra-Fredholm integral equations arising in the modeling of many applications [23].
In this section, we first describe the algorithm of the decomposition method as it applies to a general nonlinear equation of the form:

$$x(t) = a(t) + x(t) \int_0^1 \frac{t}{t+s} f(s, x(s)) ds. \tag{1}$$

The solution algorithm of equation (1) using ADM is,

$$x_0(t) = a(t), \tag{2}$$

$$x_i(t) = x_{i-1}(t) \int_0^1 \frac{t}{t+s} A_{i-1}(s) ds. \tag{3}$$

where A_i are Adomian polynomials of the nonlinear term $f(s, x(s))$, which take the form

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i \right) \right]_{\lambda=0}$$

Finally, the solution will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \tag{4}$$

IV. CONVERGENCE ANALYSIS

Theorem 4.1. Let the solution of the quadratic integral equation (1) be exist. If $|x_1(t)| < c$, c is a positive constant then the series solution (4) of the QIE (1) using ADM converges.

Proof:

Define the sequence $\{S_p\}$such that, $S_p = \sum_{i=0}^{p} x_i(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} x_i(t)$. Now,

$$f(t,x(t)) = A = \sum_{i=0}^{\infty} A_i,$$

Let S_p and S_q be two arbitrary partial sums with $p > q$. We are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space $C(I)$.

$$\begin{aligned}
S_p - S_q &= \sum_{i=0}^{p} x_i - \sum_{i=0}^{q} x_i \\
&= \sum_{i=0}^{p} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds - \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{q} A_{i-1}(s)ds \\
&= \sum_{i=0}^{p} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds - \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds \\
&+ \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds - \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{q} A_{i-1}(s)ds \\
&= \left[\sum_{i=0}^{p} x_{i-1}(t) - \sum_{i=0}^{q} x_{i-1}(t) \right] \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds \\
&+ \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \left[\sum_{i=0}^{p} A_{i-1}(s) - \sum_{i=0}^{q} A_{i-1}(s) \right] ds \\
\|S_p - S_q\| &\le \max_{t\in I} \left| \sum_{i=q+1}^{p} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=0}^{p} A_{i-1}(s)ds \right| \\
&+ \max_{t\in I} \left| \sum_{i=0}^{q} x_{i-1}(t) \int_0^1 \frac{t}{t+s} \sum_{i=q+1}^{p} A_{i-1}(s)ds \right| \\
&\le \max_{t\in I} \left| \sum_{i=q}^{p-1} x_i(t) \right| \int_0^t \frac{t}{t+s} \left| \sum_{i=0}^{p} A_{i-1}(s) \right| ds \\
&+ \max_{t\in I} \left| \sum_{i=0}^{q} x_{i-1}(t) \right| \int_0^t \frac{t}{t+s} \left| \sum_{i=q}^{p-1} A_i(s) \right| ds \\
&\le t \in I \max |S_{p-1} - S_{q-1}| \int_0^1 \frac{t}{t+s} |f(s,S_p)| ds \\
&+ \max_{t\in I} |S_q| \int_0^1 \frac{t}{t+s} |f(s,S_{p-1}) - f(s,S_{q-1})| ds \\
&\le (M + rL) \max_{t\in I} \quad |S_{p-1} - S_{q-1}| \\
&\le \alpha \|S_{p-1} - S_{q-1}\|
\end{aligned}$$

Let $p = q + 1$ then,

$$\|S_{q+1} - S_q\| \le \alpha \|S_q - S_{q-1}\| \le \alpha^2 \|S_{q-1} - S_{q-2}\| \le \cdots \le \alpha^q \|S_1 - S_0\|$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_p - S_q\| &\le \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_p - S_{p-1}\| \\
&\le [\alpha^q + \alpha^{q+1} + \cdots + \alpha^{p-1}] \|S_1 - S_0\| \\
&\le \alpha^q [1 + \alpha + \cdots + \alpha^{p-q-1}] \|S_1 - S_0\| \\
&\le \alpha^q \left[\frac{1 - \alpha^{p-q}}{1 - \alpha} \right] \|x_1\|
\end{aligned}$$

Now $0 < \alpha < 1$, and $p > q$ implies that $(1 - \alpha^{p-q}) \le 1$. Consequently,

$$\begin{aligned}
\|S_p - S_q\| &\le \frac{\alpha^q}{1 - \alpha} \|x_1\| \\
&\le \frac{\alpha^q}{1 - \alpha} \max_{t\in I} \quad |x_1(t)|
\end{aligned}$$

then $\|S_p - S_q\| \to 0$ as $q \to \infty$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space $C(I)$ and the series

$\sum_{i=0}^{\infty} x_i(t)$ converges.

V. ERROR ANALYSIS

Theorem 5.1. The maximum absolute truncation error of the series solution (4) to the QIE (1) is estimated to be,

$$\max_{t\in I}\left|x(t)-\sum_{i=0}^{q}x_i(t)\right| \le \frac{\alpha^q}{1-\alpha}\max_{t\in I}|x_1(t)|$$

Proof: From Theorem 2 we have,

$$\|S_p - S_q\| \le \frac{\alpha^q}{1-\alpha}\max_{t\in I}|x_1(t)|$$

but, $S_p = \sum_{i=0}^{p} x_i(t)$ as $p \to \infty$ then, $S_p \to x(t)$ so,

$$\|x - S_q\| \le \frac{\alpha^q}{1-\alpha}\max_{t\in I}|x_1(t)|$$

so, the maximum absolute truncation error in the interval I is,

$$\max_{t\in I}\left|x(t)-\sum_{i=0}^{q}x_i(t)\right| \le \frac{\alpha^q}{1-\alpha}\max_{t\in I}|x_1(t)|$$

and this completes the proof.

VI. NUMERICAL EXAMPLE

Example 1 Consider the following QIE,

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)x(s)ds, 5 \qquad (5)$$

This equation plays an important role in the theory of radiative transfer, neutron transport and in the kinetic theory of gases [1]. Applying ADM to equation (5), we get

$$x_0(t) = 1,$$

$$x_i(t) = x_{i-1}(t)\int_0^1 \frac{t}{t+s}\varphi(s)x_{i-1}(s)\,ds, \quad i \ge 1.$$

and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t)$$

Figure 1.a-1.d show ADM solution of problem (5) when $[\varphi(s) = 1, s, 5s^2 + 3s + 1, e^s]$.

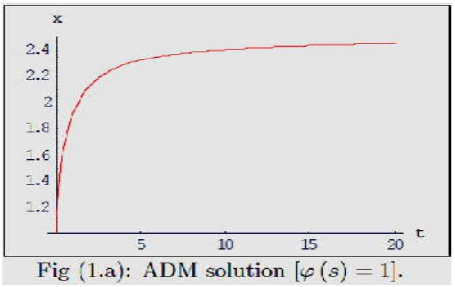


Fig (1.a): ADM solution $[\varphi(s) = 1]$.

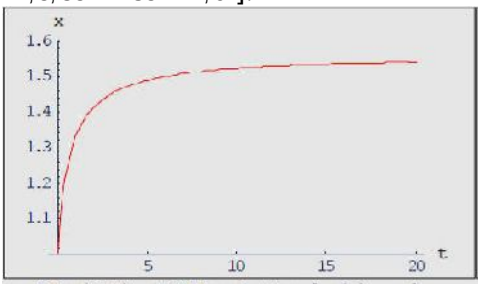


Fig (1.b): ADM solution $[\varphi(s) = s]$.

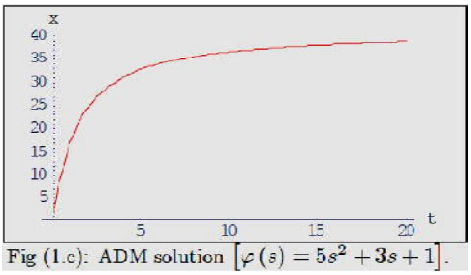


Fig (1.c): ADM solution $[\varphi(s) = 5s^2 + 3s + 1]$.

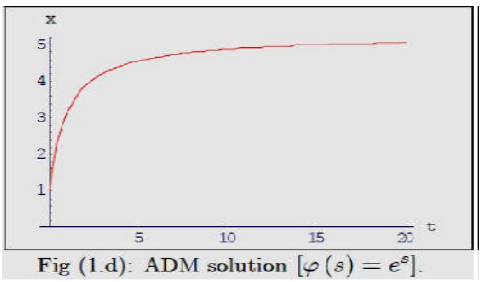


Fig (1.d): ADM solution $[\varphi(s) = e^s]$.

CONCLUSION

We have presented a convergence and error analysis for the ADM applied to the quadratic integral equation of Chandrasekhar type which appeared in the theory of radiative transfer. The numerical examples in the previous section confirmed the good convergence and error reduction as predicted by the coefficients computed with ADM and this method is efficient and easy.

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