# NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS USING MOVING LEAST SQUARES

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**Abstract—**This paper investigates numerical solution of stochastic differential equations by applying moving least squares approximation in a stochastic Galerkin projection scheme. Two different types of probability distribution functions studied here are uniform and lognormal. Numerical solution of problems with uniform random inputs are compared with Legendre-base polynomial chaos expansion. In the case of lognormal distribution, (modified) Gram-Schmidt polynomial is used for comparison. In different examples it is demonstrated that the accuracy of moving least squares is comparable with the ones of Legendre-chaos expansion and Gram-Schmidt.

## I. INTRODUCTION

An often overlooked consideration in computational mod-elling is the effect of errors or uncertainty in the input data such as system parameters, initial and boundary conditions. The reliability of computational predictions depends on assign-ing accurate values to parameters in the governing equations. In some cases, even a small uncertainty in the input data might result in non-negligible changes in the output of the system. The purpose of uncertainty quantification is therefore to investigate the effect of uncertainty in the input data and provide more reliable predictions for real-world applications.

One of the most widely used methods for uncertainty quantification is the polynomial chaos (PC) based stochas-tic Galerkin projection scheme proposed by Ghanem and Spanos [1]. In this method, a truncated version of PC ex-pansion was combined with a finite element method and successfully applied to a range of engineering problems [1], [2], [3], [4], [5]. One of the limitations of PC expansion is that the rate of convergence using Hermite basis functions is optimum (exponential) only for Gaussian random inputs. To alleviate this problem, gPC was developed in 2002 as an extension of classical PC by choosing appropriate orthogonal basis functions from the Askey scheme of hypergeometric polynomials according to the probability distribution functions (pdf) of random inputs [6]. It is numerically demonstrated in [6] and later in [7] it is proved that choosing orthogonal polynomial basis functions from the Askey scheme according to the pdfs of the input random variables leads to optimal convergence for non-Gaussian random variables. Since then gPC has been applied to many problems including random inputs with probability distributions similar to the weight function of orthogonal polynomials in the Askey scheme [8], [9], [10].
In practical problems we are often faced with distributions that lie outside the Askey scheme. In particular, positive physi-cal properties like mass, stiffness, diffusivity, etc. might follow a lognormal distribution. Therefore it is of special interest to extend the correspondence between orthogonal polynomials and pdfs to distributions outside the Askey scheme to achieve exponential convergence using optimal polynomials for such cases [11], [12], [13], [14]. In [12] multi-element gPC is developed for arbitrary distributions to generate the set of orthogonal polynomials numerically using the Stieltjes three-term recurrence procedure. The application of Gram-Schmidt orthogonalization for improving the rate of convergence for arbitrary pdfs was first proposed in [13] and subsequently applied to dynamic stall-induced fluid-structure interaction of turbine blades in [14].

Moving least squares (MLS) was first proposed in 1981 by Lancaster and Salkauskas [15] as an approximation which moving in a domain (point by point) applies a weighted least squares approximation in a support domain around each point. Years later, in 1992, Nayroles et al. [16] applied MLS to Galerkin weak-form for the first time in deterministic problems and formulated the diffuse element method (DEM). Also, a new class of deterministic meshfree schemes commonly referred to as element free galerkin (EFG) methods was developed using MLS in [17] and later applied to many engineering problems including singularity and nonlinearity [18], [19], [20]. MLS based spatial discretization schemes in combination with perturbation methods have been applied to stochastic partial differential equations in [21], [22]. More recently, some researchers have studied the application of MLS approximations in nonintrusive uncertainty quantification; see, for example, references [23], [24], [25].

## II. MOVING LEAST SQUARES

MLS is an approximation technique proposed in [15] which applies a weighted least squares approximation

to each point in a domain using the nodes in a neighbouring area around each point known as the support domain. To illustrate this approach, consider a function of random variables u(ξ) with $\xi \in D \subset R^{M}$. The domain D can be represented by a set of nodes at which the function is known. For two- and three-dimensional cases uniform sampling can be used to generate the set of nodes. Minimum discrepancy sequences such as Halton [26] and Sobol [27] can be used to generate nodes in higher dimensional stochastic spaces. After generating the set of nodes, an MLS approximation of $u(\boldsymbol{\xi})$ can be written as

$$\hat{u}(\boldsymbol{\xi}) = \sum_{k=1}^{m} p_k(\boldsymbol{\xi}) a_k(\boldsymbol{\xi}) = \boldsymbol{p}^T(\boldsymbol{\xi}) \boldsymbol{a}(\boldsymbol{\xi}), \tag{1}$$

where $m$ is the number of basis functions and $\boldsymbol{p}(\boldsymbol{\xi})$ is a vector that contains $M$-dimensional stochastic basis functions, i.e.,

$$\boldsymbol{p}(\boldsymbol{\xi}) = \begin{bmatrix} p_1(\boldsymbol{\xi}) & p_2(\boldsymbol{\xi}) & \cdots & p_m(\boldsymbol{\xi}) \end{bmatrix}^T. \tag{2}$$

The coefficient vector $\boldsymbol{a}(\boldsymbol{\xi})$ is given by

$$\boldsymbol{a}(\boldsymbol{\xi}) = \begin{bmatrix} a_1(\boldsymbol{\xi}) & a_2(\boldsymbol{\xi}) & \cdots & a_m(\boldsymbol{\xi}) \end{bmatrix}^T. \tag{3}$$

It can be noted from the above equation that the vector of coefficients is a function of the random vector $\boldsymbol{\xi}$ and can be obtained by minimizing the following functional

$$J = \sum_{i=1}^{n} W(\boldsymbol{\xi} - \boldsymbol{\xi}_i)(\boldsymbol{p}^T(\boldsymbol{\xi}_i)\boldsymbol{a}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}_i)) \tag{4}$$

where $W(\boldsymbol{\xi} - \boldsymbol{\xi}_i)$ is a bell-shaped weight function which is nonzero in a local domain around each node called the influence domain and zero outside it; and $n$ is the number of nodes in the local support domain around the point. For example, the weight function can be in the following form [28]

$$W(\boldsymbol{\xi} - \boldsymbol{\xi}_i) = \begin{cases} 1 - 6\bar{r}_i^2 + 8\bar{r}_i^3 - 3\bar{r}_i^4, & \bar{r}_i \le 1 \\ 0 & \bar{r}_i \ge 1 \end{cases} \tag{5}$$

where

$$\bar{r}_i = \frac{\|\boldsymbol{\xi} - \boldsymbol{\xi}_i\|_2}{d_s}, \tag{6}$$

in which $\|.\|_2$ denotes the Euclidean norm and $d_s$ is the radius of the support domain which is usually equal to the radius of the influence domain and is defined as $d_s = \alpha d_c$, so that $\alpha$ is the dimensionless size of the support domain and $d_c$ is the average nodal spacing in an initially estimated support domain $d_s$

$$d_c = \frac{\sqrt[M]{V_s}}{\sqrt[M]{n_{V_s}} - 1} \tag{7}$$

where $V_s$ is the volume of estimated $M$-dimensional support domain and $n_{V_s}$ is the number of nodes in that area. It should be noted that the weight function is applied around every node which is in the support domain of the point not around the point itself. This has the effect of regulating the contribution of the current point to the approximated function at nodes. Minimizing $J$ with respect to $\boldsymbol{a}$ results in the following expression for $\boldsymbol{a}$

$$\boldsymbol{a}(\boldsymbol{\xi}) = \boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{B}(\boldsymbol{\xi})\boldsymbol{U} \tag{8}$$

in which $\boldsymbol{A}(\boldsymbol{\xi})$ and $\boldsymbol{B}(\boldsymbol{\xi})$ are given by

$$\boldsymbol{A}(\boldsymbol{\xi}) = \sum_{i=1}^{n} W(\boldsymbol{\xi} - \boldsymbol{\xi}_i)\boldsymbol{p}(\boldsymbol{\xi}_i)\boldsymbol{p}^T(\boldsymbol{\xi}_i) \tag{9}$$

$$\boldsymbol{B}(\boldsymbol{\xi}) = \begin{bmatrix} W(\boldsymbol{\xi} - \boldsymbol{\xi}_1)\boldsymbol{p}(\boldsymbol{\xi}_1) & \cdots & W(\boldsymbol{\xi} - \boldsymbol{\xi}_n)\boldsymbol{p}(\boldsymbol{\xi}_n) \end{bmatrix} \tag{10}$$

and $\boldsymbol{U}$ is the vector including nodal parameters

$$\boldsymbol{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^T. \tag{11}$$

The appropriate value of $\alpha$ depends on the problem, but in general a value between 2 and 4 leads to good results [28]. In the present study, we use a uniformly distributed set of nodes and the same value of $\alpha$ all over the domain. In order to avoid singularity of $\boldsymbol{A}(\boldsymbol{\xi})$ there must be at least $m$ nodes in the local support domain which is equal to the total number of basis functions. We choose the value of $\alpha$ to be sufficiently large such that all the points on the boundaries (which are the critical points) will have at least $m$ nodes in their support domain.

Now the approximated random function in equation (1) can be written in terms of the shape functions as follows

$$\hat{u}(\boldsymbol{\xi}) = \sum_{k=1}^{n} \phi_k(\boldsymbol{\xi}) u_k = \boldsymbol{\Phi}(\boldsymbol{\xi})\boldsymbol{U}, \tag{12}$$

where $\boldsymbol{\Phi}$ is the shape function matrix given by

$$\boldsymbol{\Phi}(\boldsymbol{\xi}) = \begin{bmatrix} \phi_1(\boldsymbol{\xi}) & \cdots & \phi_n(\boldsymbol{\xi}) \end{bmatrix} = \boldsymbol{p}^T(\boldsymbol{\xi})\boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{B}(\boldsymbol{\xi}). \tag{13}$$

To avoid computing $\boldsymbol{A}^{-1}(\boldsymbol{\xi})$, the shape function matrix is rewritten as

$$\boldsymbol{\Phi}(\boldsymbol{\xi}) = \boldsymbol{\gamma}^T(\boldsymbol{\xi})\boldsymbol{B}(\boldsymbol{\xi}) \tag{14}$$

where $\boldsymbol{\gamma}(\boldsymbol{\xi})$ is the solution of the system of equations:

$$\boldsymbol{A}(\boldsymbol{\xi})\boldsymbol{\gamma}(\boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{\xi}). \tag{15}$$

If the radius of the support domain is selected to be equal to that of the influence domain, then in all the above formulations $n$ can be replaced by $N$, which is the total number of nodes in the domain. In this case the weight function in equation (4) will automatically be zero at nodes outside the support domain of the point and eventually the shape function matrix will be global with zero-element columns related to nodes outside the support domain of the point. To simplify our explanation of MLS we will use this global form in the coming formulations and replace $n$ by $N$.

## III. Stochastic Galerkin Projection Scheme

We consider a first-order system of stochastic ordinary differential equations in the general form

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}, t; \boldsymbol{\xi}), \qquad (t; \boldsymbol{\xi}) \in [0, T] \times \mathcal{D}, \tag{16}$$

$$\tag{17}$$

subject to the stochastic initial conditions

$$\boldsymbol{u}_{|_{t=0}} = \boldsymbol{g}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathcal{D}, \tag{18}$$

where $t \in [0, T]$ denotes time, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_M) \in \mathcal{D} \subset \mathbb{R}^M$ is an $M$-dimensional random vector with joint probability distribution function $\rho(\boldsymbol{\xi})$, $\boldsymbol{f} \in \mathbb{R}^n$, $\boldsymbol{g} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^n$ denotes the solution which we seek to approximate as a function of time and the random vector $\boldsymbol{\xi}$, and $n$ is the total number of

states. Theoretically, for each realization of the random vector $\boldsymbol{\xi}$ continuity of $\boldsymbol{f}(\boldsymbol{u}, t; \boldsymbol{\xi})$ for all $t \in [0, T]$ guarantees the existence of a solution to the above initial value problem. If $\boldsymbol{f}(\boldsymbol{u}, t; \boldsymbol{\xi})$ is continuously differentiable with respect to $t$ then the solution is unique.

One of the well known methods for solving this problem involves using a PC expansion for the solution $\boldsymbol{u}(t; \boldsymbol{\xi})$ combined with a stochastic Galerkin projection scheme. According to the Cameron-Martin theorem [29], any stochastic process with a finite second-order moment can be written in terms of a PC expansion. However, the convergence rate of classical Hermite PC expansion is optimal for Gaussian processes, i.e., processes in which the pdf of random variables is the weighting function used to build orthogonal Hermite polynomials. To extend the optimal convergence to some other distributions, in [6] gPC was presented as an extension of classical PC by choosing an appropriate orthogonal basis from the Askey scheme of hypergeometric polynomials [30]. The weight functions used for building orthogonal polynomials in the Askey scheme are identical to probability density functions of certain random processes. For example Hermite polynomials are associated with Gaussian distribution and Legendre polynomials are related to uniform distributions. The truncated gPC expansion of a vector of second-order (i.e., with finite variance) stochastic processes $\boldsymbol{u}(t; \boldsymbol{\xi})$ can be written as

$$\boldsymbol{u}(t; \boldsymbol{\xi}) = \sum_{k=0}^{P} \boldsymbol{a}_k(t)\psi_k(\boldsymbol{\xi}), \tag{19}$$

where $\{\psi_k\}$ is the set of gPC orthogonal basis (i.e. $\langle \psi_i \psi_j \rangle = \psi_i^2 \delta_{ij}$) up to order $p$, and $P + 1 = \frac{(M+p)!}{M!p!}$ is the total number of terms in the expansion. In non-intrusive uncertainty quantification based on gPC expansion $\boldsymbol{a}_k = \frac{\langle u\psi_k \rangle}{\langle \psi_k \psi_k \rangle}$, whereas in intrusive formulations $\boldsymbol{a}_k$ is obtained using a stochastic Galerkin projection scheme. We now outline the stochastic Galerkin projection scheme for MLS approximations and highlight the main differences between the proposed approach and the gPC based stochastic Galerkin projection scheme.

Using a MLS approximation for the solution of (16), we have

$$\boldsymbol{u}(t; \boldsymbol{\xi}) \approx \hat{\boldsymbol{u}}(t; \boldsymbol{\xi}) = \sum_{j=1}^{N} \boldsymbol{u}_j(t)\phi_j(\boldsymbol{\xi}). \tag{20}$$

Defining the expectation operator as $\langle . \rangle = \int . \rho(\boldsymbol{\xi}) d\boldsymbol{\xi}$, substituting the above approximation in equation (16), and applying the stochastic Galerkin projection scheme using MLS shape functions as weight functions results in

$$\sum_{j=1}^{N} \frac{d\boldsymbol{u}_j(t)}{dt} \langle \phi_j(\boldsymbol{\xi})\phi_i(\boldsymbol{\xi}) \rangle = \langle \boldsymbol{f}(\boldsymbol{u}, t; \boldsymbol{\xi})\phi_i(\boldsymbol{\xi}) \rangle , \tag{21}$$

where $i = 1, \ldots, N$ and this can be rewritten in the form of a first-order system of deterministic ordinary differential equations as

$$\mathbb{A} \, \frac{d\mathbb{U}(t)}{dt} = \mathbb{B}[\mathbb{U}(t)]. \tag{22}$$

When considering the $n-$dimensional state vector $\boldsymbol{u}(t; \boldsymbol{\xi}) = \begin{bmatrix} u_1(t; \boldsymbol{\xi}) & u_2(t; \boldsymbol{\xi}) & \cdots & u_n(t; \boldsymbol{\xi}) \end{bmatrix}^T$, the vector $\mathbb{U}(t)$ is given by

$$\mathbb{U}(t) = \begin{bmatrix} u_{1_1}(t) & \cdots & u_{1_N}(t) & \cdots & u_{n_1}(t) & \cdots & u_{n_N}(t) \end{bmatrix}^T \tag{23}$$

which is in $\mathbb{R}^{nN}$ and the second subscript is related to the nodal parameters in the MLS expansion of the components of $\boldsymbol{u}(t, \boldsymbol{\xi})$ (equation (20)). For $\mathbb{A}$, considering $\langle \phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}) \rangle$ as a two-dimensional tensor using a Kronecker tensor product, $\otimes$, we have

$$\mathbb{A} = \mathbb{I} \otimes \langle \phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}) \rangle , \tag{24}$$

where $\mathbb{I}$ is the $n \times n$ identity matrix. In equation (22), $\mathbb{B}[\mathbb{U}(t)] \in \mathbb{R}^{nN}$ is a vector dependent on $\mathbb{U}(t)$ and its explicit expression depends on the problem. Note that when the stochastic Galerkin projection scheme is applied using a set of orthonormal gPC basis functions, $\mathbb{A}$ becomes an identity matrix and we obtain a different expression for the vector $\mathbb{B}[\mathbb{U}(t)]$.

Similarly, the initial conditions (18) can be satisfied in a weak sense as follows:

$$\sum_{j=1}^{N} \langle \phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}) \rangle \, \boldsymbol{u}_j(0) = \langle \boldsymbol{g}(\boldsymbol{\xi})\phi_i(\boldsymbol{\xi}) \rangle \ , \quad i = 1, \ldots, N. \tag{25}$$

The preceding equation can be rewritten in the form of a linear system of equations as

$$\mathbb{A} \, \mathbb{U}(0) = \mathbb{D}, \tag{26}$$

where $\mathbb{D}$ can be written using Kronecker product notation (considering $\phi_i(\boldsymbol{\xi})$ as a one-dimensional tensor) as follows

$$\mathbb{D} = \langle \boldsymbol{g}(\boldsymbol{\xi}) \otimes \phi_i(\boldsymbol{\xi}) \rangle. \tag{27}$$

As shown in equations (22) and (26), after applying the stochastic Galerkin projection scheme to stochastic ordinary differential equations of the form (16) and (18) we obtain a system of deterministic ordinary differential equations which can be solved numerically to find the deterministic coefficients in the stochastic expansion of the response (20). In the present study a fourth-order Runge-Kutta scheme is used to solve the system of ODEs.

## IV. Post-processing

Consider a stochastic process $u(t; \boldsymbol{\xi})$ approximated using MLS as $\hat{u}(t; \boldsymbol{\xi}) = \sum_{j=1}^{N} u_j(t)\phi_j(\boldsymbol{\xi})$. The mean of the stochastic process can hence be approximated as

$$\mu(t) = \sum_{j=1}^{N} u_j(t) \langle \phi_j(\boldsymbol{\xi}) \rangle. \tag{28}$$

Also, considering the fact that MLS shape functions are not orthogonal we obtain the following expression for the variance approximation:

$$Var(t) = \sum_{j=1}^{N} \sum_{i=1}^{N} u_i(t)u_j(t) \big( \langle \phi_i(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}) \rangle - \langle \phi_i(\boldsymbol{\xi}) \rangle \langle \phi_j(\boldsymbol{\xi}) \rangle \big). \tag{29}$$

## V. Numerical Results

In engineering and science, discretizing a system of time-dependent stochastic partial differential equations (SPDEs) results in a non-linear or linear system of stochastic differential equations (SODEs). That is why SODEs are used as model equations for developing uncertainty quantification methods. There are three different examples studied in this paper. First, a one-dimensional stochastic decay problem is used to study the basic convergence properties of the MLS-based method suggested here. Then, the Kraichnan-Orszag three mode problem is used to study a three-dimensional case including stochastic singularity. Finally a spring-mass example with random stiffness is used to investigate the accuracy of the proposed approach for problems with arbitrary random inputs. MLS parameters used for one-dimensional examples are $p = 1$ and $\alpha = 1.5$ for $N \leq 3$, $p = 2$ and $\alpha = 2.7$ for $N = 4$, $p = 3$ and $\alpha = 3.2$ for $N \geq 5$. For the three-dimensional example a circular support domain is used with $\alpha = 2.5$ and the order of polynomial basis is $p = 2$. A higher-order Gauss-Legendre quadrature is used for computing MLS arrays where the number of quadrature points in each direction is three times the number of nodes in that direction.

### A. The decay problem

In this section we apply the proposed stochastic MLS Galerkin projection scheme to the decay problem and compare the accuracy and convergence with gPC. The performance of gPC for this problem has been studied earlier by Xiu and Karniadakis [6]. This governing equation and initial condition are given by

$$\frac{du}{dt} = -ku, \qquad u(0) = u_0, \tag{30}$$

where $k = \xi$ is a uniformly distributed random variable, i.e., $\xi \sim U[-1, 1]$ and $\rho_\xi(\xi) = \frac{1}{2}$. The exact solution of the equation is given by

$$u(t;\xi) = u_0 e^{-\xi t}. \tag{31}$$

The mean and variance of the exact solution can hence be written as

$$\mu_{exact}(t) = \frac{u_0}{2t}(e^t - e^{-t}), \tag{32}$$

$$Var_{exact}(t) = u_0^2 \left[\frac{e^{2t} - e^{-2t}}{4t} - \frac{(e^{2t} - e^{-2t})^2}{4t^2}\right]. \tag{33}$$

Figure (1) shows how accuracy of MLS is increased in terms of the number of nodes. It is seen that for the same order of polynomial basis in MLS and gPC, if there are more nodes used in MLS it will be more accurate.

Figure (2) compares the rate of convergence in gPC and MLS for the same number of terms in their expansions. Superscript $*$ for MLS denotes the case that the quadrature points selected for MLS are the same as gPC.

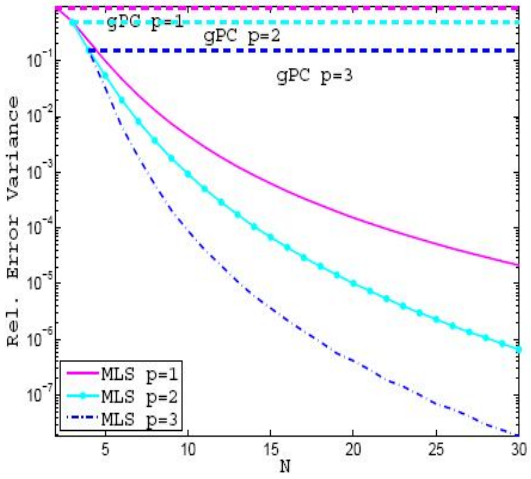


Fig. 1: Relative error in variance of the solution at $t = 5$ computed using MLS as a function of the number of nodes for different orders of polynomial basis compared with gPC for the same order of basis.

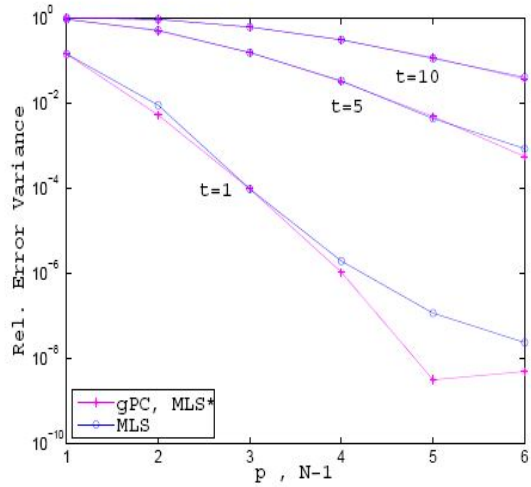


Fig. 2: Comparison of the convergence of MLS with increasing values of $N$ and gPC with increasing values of $p$. MLS* is the case where the integration points in MLS and gPC are the same

### B. The Kraichnan-Orszag three mode problem

A well-known three-dimensional benchmark problem including a stochastic singularity which is proposed in [31] and studied in [32] and [33] is the Kraichnan-Orszag three mode problem given below:

$$\begin{aligned} \frac{dx}{dt} &= yz, \\ \frac{dy}{dt} &= xz, \\ \frac{dz}{dt} &= -2xy, \end{aligned} \tag{34}$$

subject to stochastic initial conditions

$$x(0) = x_0(\xi), \qquad y(0) = y_0(\xi), \qquad z(0) = z_0(\xi). \tag{35}$$

In general, the response of the system is periodic unless $x(0) = \pm y(0)$ in which case a singularity occurs and the period of the system goes to infinity. When the stochastic inputs are such that the condition $x(0) = \pm y(0)$ is met, then the gPC approximation deviates from the exact solution after a relatively short amount of time has elapsed.

To investigate three-dimensional random inputs we consider the following initial conditions

$$x(0) = \eta + 0.01\xi_1, \quad y(0) = \beta + 0.01\xi_2, \quad z(0) = \gamma + 0.01\xi_3, \quad (36)$$

where $\xi_1, \xi_2, \xi_3 \sim U[-1, 1]$ are independent random variables and we set $\beta = \gamma = 1$. Figure (3) and (4) show the variance results for MLS and gPC (compared with Monte Carlo as exact solution with the indicated number of samples) for non-singular and singular cases, respectively. It is seen that MLS is as accurate as gPC, and for the singular case it deviates from the exact solution at an earlier time compared to the non-singular case.

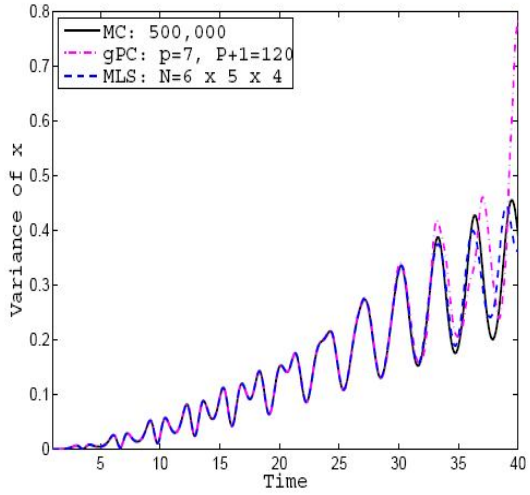


Fig. 3: Comparison of mean and variance for the three-dimensional non-singular case where $\eta = 0.94$.

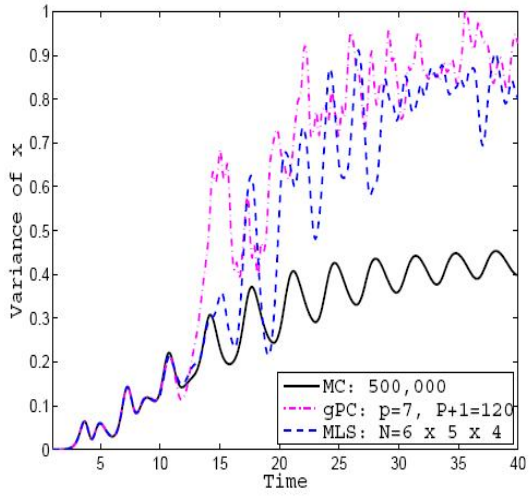


Fig. 4: Comparison of mean and variance for the three-dimensional singular case where $\eta = 0.98$

### C. Spring mass system with arbitrary uncertain stiffness

Now, consider a spring mass system with the following equation of motion (second-order ODE) and initial conditions

$$m\ddot{x} + kx = 0, \quad x(0) = 1, \ \dot{x}(0) = 0, \quad (37)$$

where $x$ is the displacement measured from equilibrium point, $m$ is mass (in the numerical studies, we set $m = 1$) and $k = \xi \in [a, b]$ is the stiffness following lognormal distribution $\rho_\xi(\xi)$. Considering $\mu$ as mean and $\sigma$ as standard deviation (square root of variance), the lognormal probability distribution function is defined as

$$\rho_\xi(\xi; \mu, \sigma) = \frac{1}{\xi\sigma\sqrt{2\pi}} \exp\Big(-\frac{(\ln\xi - \mu)^2}{2\sigma^2}\Big), \quad \xi > 0, \quad (38)$$

where $\mu = 2.0$ and $\sigma = 0.5$ for this example.

MLS results are compared with the ones obtained from modified Gram-Schmidt (MGS). It is seen in Figure (5) and (6) that with increasing the number of terms, MLS results will be accurate for a longer period of time while MGS eventually fails to converge because of instability of orthogonalization for creating high order polynomial basis functions.

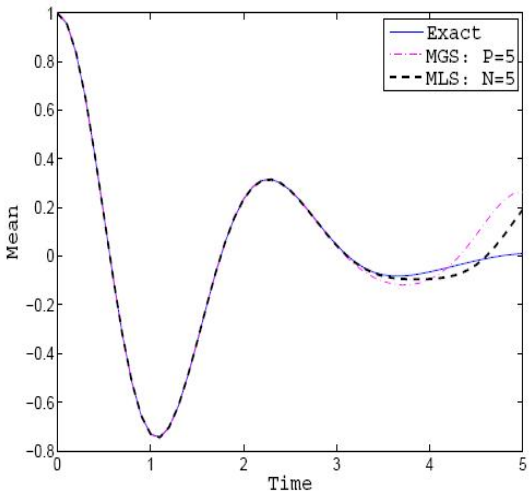


Fig. 5: Comparison of the mean of the MGS and gPC with 5 terms in their expansions.

### D. Conclusion

An MLS-based intrusive uncertainty quantification method was developed in this paper. It was shown in one- and three-dimensional examples that it is as accurate as gPC for uniform type of pdf. For the case of arbitrary distribution, orthogonalization methods fail for large number of basis functions because of instability in creating higher order polynomials. In comparison, MLS keeps converging with increasing the number of terms (nodes) in the approximation. For the case of stochastic singularity both MLS and MGS deviate from the exact solution at earlier times and this can be considered as the future step of developing an MLS-based uncertainty quantification scheme.

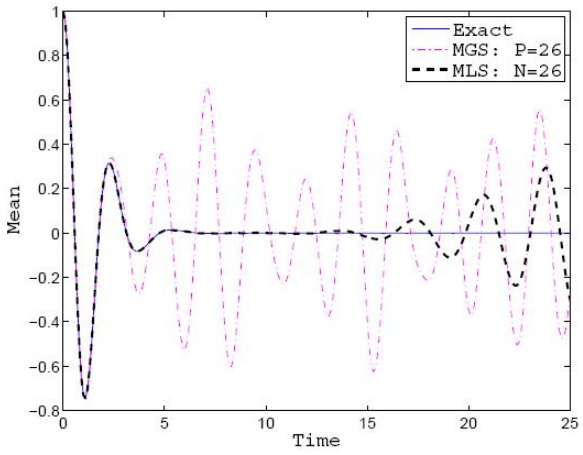


**Fig. 6: Comparison of the mean of the MGS and gPC with 26 terms in their expansions.**

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