HELICITY AS ISOTOPY INVARIANT

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Abstract- In this paper we will investigate whether or not the writhe of knot is an isotopy invariant? By adding a parallel strand to an oriented knot/link, we will justify that writhe is a useful isotopy invariant. Since helicity has its origin in knot theory, consequently, we will eatablish that helicity is an isotopy invariant which will be useful to study plasma physics, solar physics and astrophysics.

Keywords- Writh, Isotopy Invariant, Magnetic Field, Reidemeister Moves, Helicity.

I. INTRODUCTION

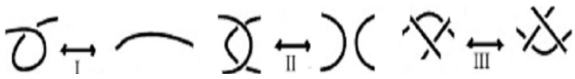
Mathematicians were perplexed at the seemingly unending number of ways a knot could be shaped and turned. Consequently, these give rise to the central problem of knot theory i.e., whether two knots (links) are equivalent or not. This was the motivation for much of the recent work in knot theory, which is devoted to search for invariants of knots. The study of invariants underwent in a kind of phase transition, which has linked knot theory to chemistry, molecular chemistry, mathematical physics, particles physics, polymer physics, statistical mechanics, fluid mechanics, kinematics, C*-algebra, conformal field theory, crystallography, cryptography, graph theory, computer systems and networks, etc.

In the recent past, biologists and chemists studying genetics discovered an exciting link of knot theory with DNA (genetic material of all cells, containing coded information about cellular molecules and processes) and synthetic chemistry. DNA is just one application of knot theory, which presently is an area of intense mathematical activities worldwide. The relation between knots (links) and graphs has provided a new way of visualizing, redefining and deriving many concepts in knot theory. One can enjoy transferring some qualitative information between knot theory and graph theory. Plasma Physicists study helicity to solve complex problems using useful skills from knot theory. In this study we will investigate an analogue property of helicity from knot theory perspective. We will investigate whether or not the writhe of knot is an isotopy invariant? By adding a parallel strand to an oriented knot/link, we will justify that writhe is a useful isotopy invariant. Since helicity has its origin in knot theory, consequently, we will eatablish that helicity is an isotopy invariant which will be useful to study plasma physics, solar physics and astrophysics.

II. MATERIAL AND METHODS

Homotopy is a continuous deformation of an object which preserves its topological integrity. Isotopy is a continuous deformation, i.e., Rubber-Sheet Geometry wherein bending, twisting, stretching and pulling is allowed but no self-intersections and cutting. A knot is an isotopy class of embedding of a unit circle in R 3.or one may consider as piecewise linear, simple closed curve in R3

. Two knots are equivalent if one can be transformed into the other via a deformation of R3 upon itself (known as an ambient isotopy), equivalently two knots are equivalent (via Reidmeister moves) iff (any of) their projections differ by a finite sequence of Reidemeister moves. By the planar isotopy we mean the motion of the projection in the plane that preserves the graphical structure of the underlying universe. The pivotal moves in the theory of knots are the Reidemeister moves. We will view these moves as Reidemeister moves of type I, II, and III.



Ambient isotopy and equivalence via Reidemeister moves is the same. In the future, knots will be confused with their class of projections with crossings indicated unless otherwise stated.

In knot theory, the writhe is a property of an oriented link diagram. The writhe is the total number of positive crossings minus the total number of negative crossings.

A direction is assigned to the link at a point in each component and this direction is followed all the way around each component. If as you travel along a link component and cross over a crossing, the strand underneath goes from right to left, the crossing is positive; if the lower strand goes from left to right, the crossing is negative. One way of remembering this is to use a variation of the right-hand rule.



+1 -1

For a knot diagram, using the right-hand rule with either orientation gives the same result, so the writhe is well-defined on un-oriented knot diagrams. Linking number, which is the number of times the two curves are intertwined. It is a topological invariant, meaning that amount of stretching or pulling will not change the linking number without cutting and reconnecting the strands. Twist, which is most easily imagined as the number of times each of the curves rotates around the central axis of the double helix. This should not be confused with linking number. Writhe, which loosely put is the number of times the central axis makes loops about itself. A very interesting equation referred to as White's Formula sums up the relationship between these quantities quite simply:
Linking Number = Twist +Writhe. Linking Number, Twist, and Writhe as Integers.

III. RESULTS AND DISCUSSION

The results and discussion in this article are variational, diagrammatic and illustrative, suspecting an easy understanding of writhe properties ans consequently helicity. The writhe of a knot is preserved by Reidemeister moves of Type II and Type III but Reidemeister move of Type I increases or decreases the writhe by 1. Consequently, writhe of a knot is not an isotopy invariant of the knot. Below is a discussion, how Reidemeister move of type I can be modified to play an important role in study of writhe..
Type II Move:

$\omega_r\{$  $\} = 0$ and $\omega_r\{$  $\} = \{+1-1\}=0$

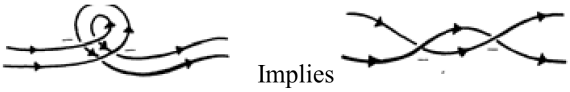
Type III Move:

$\omega_r\{$  $\} = +1$ (or -1) and $\omega_r\{$  $\} = +1$ (or -1)

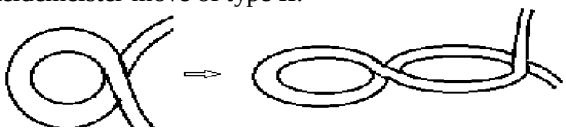
Azram [4] has established that a 2π-twist and/or a π-twist move can be established by applying sequence of Reidemeister moves on parallel strand.



Writhe of a knot is not invariant under Type I move because it changes by ± 1. For a knot K, we can obtained a link K* by adding a parallel strand. In case of same orientation writhe of K* is an invariant under type I move. Below, see the diagrammatic illustration which is consequence of 2π-twist.



Consequently, Reidemeister move of type I followed by 2π-twist to crossing conversion is equivalent to Reidemeister move of type II.



The above discussion leads to a useful observation to study magnetic flux and consequently helicity.

Helicity is defined as $H = \int \vec{A} \bullet \vec{B}\, dv$ where $\vec{B}$ stands for magnetic field and $\vec{A}$ stand for magnetic vector potential and are related as $\vec{B} = curl(\vec{A})$. Since helicity is related with linked tubes so it give connection with knot theory in term of knottedness, twistedness and linkedness. To study helicity, helicity meter is used to measure the square of magnetic flux which is read exactly as we have introduced crossing sign above with output as $\pm + \Phi^2$ where as $\Phi = \int \vec{B} \bullet d\vec{S}$ with $d\vec{S}$ as normal vector to surface area element dS^3. Following is an analoque result of [5,6]

Theorem Helicity is isotopy invariant in a magnetic flux strand.

Proof
Type II Move:
Reidemeister move of Type I in a magnetic flux strand gain a crossing and consequently adding magnetic flux of $\pm \Phi^2$ but at the same time there a twist which contribute magnetic flux $\mp \Phi^2$

Magnetic Flux {  } implies Magnetic Flux {  } $= \pm \Phi^2$ implies

Magnetic Flux {  } $= \mp \Phi^2$

Hence, H {  } = H {  }

Type II Move:

Magnetic Flux {  } = 0 and Magnetic Flux {  } = $\{+\Phi^2 - \Phi^2\}$=0 implies

H{  } = H{ }=0

Type III Move:

Magnetic Flux {  } $= +\Phi^2$(or $-\Phi^2$) and Magnetic Flux {  } $= +\Phi^2$(or $-\Phi^2$)

Hence, H {  } = H {  }

This completes the proof■

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