SOME INTERESTING PROPERTIES OF TWO SPECIAL TYPES OF TRIDIAGONAL MATRICES

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Abstract- In this work, we discuss two articles which Jonas Rimas published in 2005. We found that they have some interesting properties of two special types of tridiagonal matrices. These properties serve the purpose of computing the eigenvalues and eigenvectors of this type of matrices, and consequently in computing integer powers of such matrices.1991 Mathematics Subject Classification.65Q05; 39A05.

Key words- and phrases. difference equations; tri diagonal matrices; orthogonal polynomials; powers and inverses.

I. INTRODUCTION

J. Rimas in [3,4] discussed computing positive integer powers of a special type of tridiagonal matrices of size $k \times k$, these matrices are of the form:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & .. & .. & .. & 1 & 0 \end{bmatrix} \quad (1)$$

and in [2], we introduced further interesting properties of these matrices. In particular, we derived and proved special properties of the characteristic polynomials of these matrices - to be mentioned in the sequel-. Such properties play an important role in reduction of the computations of the eigenvalues, which in turn is an important component in the computation of integer powers of these matrices.

In the present work, we discuss another special type of tridiagonal matrices of size $k \times k$ which J. Rimas introduced in [5], these matrices are of the form:

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & .. & .. & .. & 1 & 1 \end{bmatrix} \quad (2)$$

We found that matrices of the form A_2 have a very close relation with matrices of the form A_1. The characteristic polynomial of A_2 is equal to a linear factor multiplied by the characteristic polynomial of A_1 of size one less than the size of A_2. Although we have confidence that this relation is true, but it seems that the proof needs more work. Therefore, we are going to present this as a conjecture in the present article, and we hope to prove it very soon. This conjecture means that if A_2 has size kxk, then $(k-1)$ of its eigenvalues are exactly the eigenvalues of A_1 of size $(k-1) \times (k-1)$. When this is added to the properties of the characteristic polynomial of A_1, which we proved in [2], it makes the computation of these eigenvalues much easier and less costly.

Section 2 presents some mathematical preliminaries relevant to the subject. Section 3 includes the main results. Section 4 discusses the consequences of these results, and what can be concluded from them.

II. MATHEMATICAL PRELIMINARIES

It is known that tridiagonal matrices, orthogonal polynomials, and second-order linear homogeneous difference equations are very much related. Given the tridagonal matrix

$$\begin{bmatrix} \frac{x-a_1}{b_1} & 1 & 0 & \dots & 0 \\ \frac{c_2}{b_2} & \frac{x-a_2}{b_2} & 1 & 0\dots & 0 \\ 0 & \frac{c_3}{b_3} & \frac{x-a_3}{b_3} & 1\dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ & & 0 & \frac{c_n}{b_n} & \frac{x-a_n}{b_n} \end{bmatrix} \quad (3)$$

where $a_n \geq 0, b_n > 0, c_n > 0, for all n \geq 0$, and $c_1 = b_n = 1$. The second-order linear homogeneous difference equation

$$p_n(x) = \frac{x - a_n}{b_n} p_{n-1}(x) - \frac{c_n}{b_n} p_{n-2(x),} n \geq 2 \quad (4)$$

with initial conditions: $p_0(x) = 1, p_1(x) = \frac{x-a_1}{b_1}$, generates the set of orthogonal polynomials $\{p_k(x)\}_{k=2}^{n}$. It is known that each polynomial
$p_k(x), 1 \leq k \leq n$, is the determinant of the principal $k \times k$ submatrix of the matrix in (3) above.

In [1], we showed that the polynomial $p_n(x)$, generated using the recursion relation in (4), is the characteristic polynomial of the tridiagonal matrix:

$$T = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_2 & a_2 & b_2 & 0 & \\ 0 & c_3 & a_3 & b_3 & \\ \dots & \dots & \dots & \dots & \dots \\ & & & c_n & a_n \end{bmatrix} (5)$$

and that this matrix T is equal to $D^{-1}A$, where $D = Diagonal(\frac{1}{b_1}, \frac{1}{b_2}, \dots, \frac{1}{b_{n-1}}, 1)$, and

$$A = \begin{bmatrix} \frac{a_1}{b_1} & 1 & 0 & \dots & 0 \\ \frac{c_2}{b_2} & \frac{a_2}{b_2} & 1 & 0 \dots & 0 \\ 0 & \frac{c_3}{b_3} & \frac{a_3}{b_3} & 1 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ & & 0 & \frac{c_n}{b_n} & \frac{a_n}{b_n} \end{bmatrix} (6)$$

The polynomial $p_n(x)$ is the characteristic polynomial of the tridiagonal matrix T. So, the roots of $p_n(x)$ are the eigenvalues of the matrix T, say
$x_1, x_2, \dots, x_n$, and for each $x_i, i = 1, \dots, n$, the corresponding eigenvector is:

$$p_0(x_i), p_1(x_i), \dots, p_{n-1}(x_i)]^T (7)$$

Throughout the remaining sections, we will denote the characteristic polynomial of matrices of the form A_1 in (1) by $\{p_k(x)\}_{k=0}^{n}$, and the characteristic polynomial of matrices of the form A_2 in (2) by $\{q_k(x)\}_{k=0}^{n}$.

III. MAIN RESULTS

In our paper [2], we showed that matrices of the form A_1 in (1) enjoy the following properties:
1. For each $k > 1$, $p_k = p_{\frac{k}{2}}^2 - p_{\frac{k}{2}-1}^2$, when k is even, and $p_k = p_{\frac{k-1}{2}}(p_{\frac{k-1}{2}+1} - p_{\frac{k-1}{2}-1})$. When k is odd.
2. For each $k 0, p_k(x)$ is even when k is even, and $p_k(x)$ is odd when k is odd.
It is worth to mention at this point that since both matrices A_1 and A_2 are symmetric, all the eigenvalues are real. It is also worth to notice that the specially structured $n \times n$ tridiagonal matrices which Rimas discussed in [3,4] are of the form in (1) and (2) above. So, $a_i = 0, 1 \leq i \leq n$, and $b_i = c_i = 1, 1 \leq i \leq n$. The second-order linear homogeneous difference equation (3) in this case has the form

$$p_k(x) = p_1(x)p_{k-1}(x) - p_{k-2}(x), k \geq 2 (7)$$

with $p_0(x) = 1, p_1(x) = x$.

Lemma 1 $\lambda = 2$ is always an eigenvalue of the matrix A_2 in (2).
Proof. The row-echelon form of $A_2 - 2I$ has always a bottom row of zeros. (I is the identity matrix of the same size as A_2).

Lemma 2 $\lambda = 2$ is the dominant eigenvalue of A_2 in (2).
Proof. By direct application of Gershgorin Circle theorem.

Conjecture 3 $q_k(x) = (x - 2)p_{k-1}(x)$.
Now, since $\lambda = 2$ is the dominant eigenvalue of the matrix A_2, we can use the well known deflation technique in

numerical analysis to obtain a matrix B of size $(k-1)x(k-1)$ which has the remaining eigenvalues of A2 except $\lambda = 2$. To find the corresponding eigenvector, we have to solve the homogeneous system $(A2-2I)x = 0$ for x. This implies that the eigenspace is one-dimensional with basis $(1,1,1,1)^T$. Deflation is done using the formula $B = A_2 - \frac{1}{x_{11}}.(x_1.a_1)$, where x_{11} is the first component of the eigenvector $x_1 = (1,1,1,1)^T$, and a_1 is the first row of A_2. So, we have,

$$\begin{bmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & .. & .. & .. & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & .. & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ .. & .. & 0 & 0 & .. & 1 \\ -1 & -1 & 0. & .. & 1 & 1 \end{bmatrix}$$

The matrix B is obtained from this resulting matrix by deleting the first row and the first column, to get

$$B = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ .. & 0 & 0 & .. & 1 \\ -1 & 0. & .. & 1 & 1 \end{bmatrix}$$

As we mentioned above, the matrix B has size $(k-1)\times(k-1)$, and has the same eigenvalues as A2 except $\lambda = 2$.

It is known that the characteristic polynomial of a matrix of size $k \times k$ is a monic polynomial of degree k. So, if $q_k(x)$ is the characteristic polynomial of the matrix A_2, and r_{k-1} is the characteristic polynomial of the deflated matrix B, and if

$q_k(x) = x^k + a_{k-1}x^{k-1} + \dots + a_1x + a_0$, and if $r_{k-1}(x) = x^{k-1} + b_{k-2}x^{k-2} + \dots + b_1x + b_0$, then we have

$$q_k(x) = (x-2).r_{k-1}(x), \text{ and}$$

$$q_k(x) = x^k + a_{k-1}x^{k-1} + :: + a_1x + a_0 = (x-2).(x^{k-1} + b_{k-2}x^{k-2} + \dots + b_1x + b_0)$$

By comparing coefficients, we get the following:

$$\begin{aligned} 2b_0 &= a_0 \\ b_0 - 2b_1 &= a_1 \\ b_1 - 2b_2 &= a_2 \\ &\vdots \\ b_{k-2} - 2b_{k-1} &= a_{k-1}, \text{where} b_{k-1} = 1. \end{aligned}$$

And this yields the following:

$$b_0 = -\frac{a_0}{2}$$

$$b_1 = \frac{1}{2}(b_0 - a_1)$$

$$b_2 = \frac{1}{2}(b_1 - a_2)$$

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$$b_{k-2} = \frac{1}{2}(b_{k-3} - a_{k-2})$$

$$b_{k-1} = \frac{1}{2}(b_{k-2} - a_{k-1}) \Leftrightarrow b_{k-2} - a_{k-1} = 2$$

IV. COMPUTATIONAL CONSIDERATIONS

4.1 When A_2 has order $2n \times 2n$

To find the eigenvalues of A_2; we need to find the roots of the characteristic polynomial of A_2, $q_{2n}(x)$, but we have, $q_{2n}(x) = (x-2)p_{2n-1}(x)$. This reduces to finding the roots of $p_{2n-1}(x)$, because $\lambda = 2$ is always an eigenvalue of A_2.

But $p_{2n-1}(x) = p_{n-1}(x)(p_n(x) - p_{n-2}(x))$: So, we have to solve,

(1)$p_{n-1}(x) = 0$, this gives $(n-1)$ roots, say $x_1; x_2; \dots; x_{n-1}$.

(2) $p_n(x) - p_{n-2}(x) = 0$, this gives n roots, say $x_n; x_{n+1}; \ldots; x_{2n-1}$.
So, the $2n$ eigenvalues of A_2 are $x_1; x_2; \ldots; x_{n-1}, x_n; x_{n+1}; \ldots; x_{2n-1}$.

4.2 When A_2 has order $(2n + 1) \times (2n + 1)$

To find the eigenvalues of A_2, we need to find the roots of the characteristic polynomial of A_2; $q_{2n+1}(x)$, but we have, $q_{2n+1}(x) = (x - 2)p_{2n}(x)$, this reduces to finding the roots of $p_{2n}(x)$; and we know that $p_{2n}(x) = p_n^2(x) - p_{n-1}^2(x) = [p_n(x) - p_{n-1}(x)][p_n(x) + p_{n-1}(x)$
So, we have to solve,
(1) $p_n(x) = p_{n-1}(x)$, and this gives n eigenvalues, say .x1; x2; ... ; xn
(2) $p_n(x) = -p_{n-1}(x)$, and this gives -x1; $-x2; \ldots; -xn$.
So, the $(2n + 1)$ eigenvalues of A_2 are: $\pm x1; \pm x2; \ldots; \pm xn$.

Example 4- Consider the matrix $A_2$with size 5×5, the characteristic polynomial of A_2 is:

$q_5(x) = x^5 - 2x^4 - 3x^3 + 6x^2 + x - 2$, So we have: $a_0 = -2; a_1 = 1; a_2 = 6; a_3 = -3; a_4 = -2;$ and this gives:
$b_0 = -\frac{-2}{2} = 1, b_1 = \frac{1}{2}(1 - 1) = 0, b_2 = \frac{1}{2}(0 - 6) = -3, b_3 = \frac{1}{2}(-3 - (-3)) = 0, b_4 = \frac{1}{2}(0 - (-2)) = 1.$
Thus,

$$r_4(x) = x^4 - 3x^2 + 1.$$

Remark 5- Notice that $r_4(x) = p_4(x)$, which is the characteristic polynomial of the 4×4 matrix of the form A_1.
Example 6- Consider the matrix

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

of size 6×6.This matrix has the characteristic polynomial
$q_6(x) = x^6 - 2x^5 - 4x^4 + 8x^3 + 3x2 - 6x = (x - 2)(x^5 - 4x^3 + 3x) = (x - 2)p_5(x)$, where $p_5(x) = x^5 - 4x3 + 3x$ is the characteristic polynomial of the matrix

$$A_1 = [[c]lllll0100010100010100010100010],$$

the eigenvalues of A_2 are the eigenvalues ofA_1 with $\lambda = 2$ added to them.But the eigenvalues of A_1 are the roots of $p_5(x)$, and these roots are: $\pm 1, 0, \pm\sqrt{3}$. So, the eigenvalues of A_2 are: $\pm 1, 0, \pm\sqrt{3}, 2$.

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